# MATH 101 - SETS, GROUPS, AND TOPOLOGY, FALL 2018
## ASSIGNMENT 12

**Due Friday, October 26 at the beginning of class** (please submit your assignment on Canvas). Make sure to include your full name *and the list of your collaborators* (if any) with your assignment. You may discuss problems with others, but you may *not* keep a written record of your discussions. Please refer to the syllabus for details.

In your proofs, you may assume basic facts about real numbers and integers, such as those mentioned on p. 90-91 of Hammack. You also can use results from the course textbooks, results covered in class, or results you have proven already. As a general rule, you should imagine that you are writing your proof to convince somebody else in the class who is very skeptical about the particular statement. In particular, it should be completely understandable to another student: always justify your reasoning in plain English.

(1) Recall that the *order* of an element $a$ in a group $G$ is the minimal natural number $n$ (if it exists) such that $a^n = e$.
  (a) Give the the order of each element in $\mathbb{Z}_8$ (give a brief justification).
  (b) Give the order of each element in $D_4$, the group of symmetries of a square (give a brief justification).

(2) Assume $f : G \to H$ is an isomorphism. Prove that if $G$ is cyclic, then $H$ is cyclic.

(3) Prove that $\mathbb{Z}_2 \times \mathbb{Z}_3$ is cyclic, but $\mathbb{Z}_2 \times \mathbb{Z}_4$ is not cyclic.

(4) Assume that $G$ is a finite group and $a \in G$.
  (a) Prove that there exists a (strictly positive) natural number $n$ such that $a^n = e$.
  (b) Prove that the order of $a$ is equal to the order of $a^{-1}$.

(5) Prove the following:
  (a) If $G$ is a group and $a \in G$ has order $n$, then $\langle a \rangle = \{a^0, a^1, \ldots, a^{n-1}\}$, and all those powers are distinct. *(Recall that $\langle a \rangle$ is defined to be the set of all integer powers of $a$.)*
  (b) If $G$ is an infinite cyclic group, then $\mathbb{Z}$ is isomorphic to $G$.
  (c) If $G$ is a finite cyclic group with $n$ elements, then $\mathbb{Z}_n$ is isomorphic to $G$.

(6) Assume $G$ is group and $H$ is a subgroup of $G$. We define a relation $\sim$ on $G$ by: $a \sim b$ if there exists $x \in H$ such that $a = bx$.
  (a) Prove that $\sim$ is an equivalence relation.
  (b) Write the equivalence classes of $\sim$ if $G = \mathbb{Z}_6$ and $H = \{[0], [3]\}$ (the operation on $G$ is addition modulo 6).

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*Date*: October 22, 2018.