MATH 101 - SETS, GROUPS, AND TOPOLOGY, FALL 2018
ASSIGNMENT 2

Due Friday, September 14 at the beginning of class (please submit your assignment on Canvas). Make sure to include your full name *and the list of your collaborators* (if any) with your assignment. You may discuss problems with others, but you may *not* keep a written record of your discussions. Please refer to the syllabus for details.

In your proofs, you may assume basic facts about real numbers and integers, such as those mentioned on p. 90-91 of Hammack. You also can use results covered in class or results you have proven already. As a general rule, you should imagine that you are writing your proof to convince somebody else in the class who is very skeptical about the particular statement. In particular, it should be completely understandable to another student: always justify your reasoning in plain English.

(1) Assume that x and y are integers. Prove the following:
  (a) If both x and y are odd, then $x + y$ is even.
  (b) If $7xy$ is even, then either x is even or y is even.
  (c) If x is a multiple of 3, then $x + 1$ is *not* a multiple of 3.
  (d) If $x^2 + 1$ is a multiple of 3, then x is *not* a multiple of 3. *Hint: you are always allowed to use statements that have been proven previously.*

(2) The *absolute value* $|x|$ of a real number x is defined by:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Using this definition, prove the following basic properties of the absolute value:
  (a) If x is a real number, then $x \leq |x|$.
  (b) If x and y are real numbers, then $|xy| = |x||y|$.

(3) For x and b both strictly positive real numbers, $\log_b(x)$ is defined to be[1] the number y such that $b^y = x$. Prove that $\log_2(3)$ is irrational.

(4) Rewrite each of the statements below to have the form "If P then Q". Then decide whether the statement is true or false. If it is true, give a proof. If it is false, give an explicit counterexample (and *justify* why your counterexample works). The first one has been done for you as an example.
  (a) The product of two odd integers is a multiple of 3. **Solution.** This can be rewritten as "If x and y are both odd integers, then xy is a multiple of 3". This is a *false* statement: $x = 1$ and $y = 5$ are both odd, but $xy = 5$ is not a multiple of 3.
  (b) The sum of two rational numbers is rational.
  (c) The product of two irrationals is irrational.
  (d) The product of an irrational and a nonzero rational is irrational.
  (e) The sum of two irrational numbers is irrational.

Date: September 7, 2018.

[1] You may take it as a given that this number exists and is unique.