MATH 101 - SETS, GROUPS, AND TOPOLOGY, FALL 2018
ASSIGNMENT 4

Due Friday, September 20 at the beginning of class (please submit your assignment on Canvas). Make sure to include your full name *and the list of your collaborators* (if any) with your assignment. You may discuss problems with others, but you may *not* keep a written record of your discussions. Please refer to the syllabus for details.

In your proofs, you may assume basic facts about real numbers and integers, such as those mentioned on p. 90-91 of Hammack. You also can use results covered in class or results you have proven already. As a general rule, you should imagine that you are writing your proof to convince somebody else in the class who is very skeptical about the particular statement. In particular, it should be completely understandable to another student: always justify your reasoning in plain English.

(1) Show that for any non-negative integer n, $n < 2^n$.

(2) Prove using induction that for any real number $r \neq 1$ and any non-negative integer n, $\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$.

(3) For n a non-negative integer, the nth *Fibonacci number*, F_n, is defined recursively by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ (you can read more about the Fibonacci numbers in Section 10.3 of Hammack). Prove that for any non-negative integer $n \geq 0$, $\sum_{i=0}^{n} F_i^2 = F_n F_{n+1}$.

(4) (a) What is wrong with the following proof that all books have the same title?

We prove by induction on $n \geq 1$ the statement S_n: in any collection $b_1, b_2, \dots, b_n$ of n books, all books have the same title.

- For the base case, consider S_1: any one book has the same title. This is clearly true since there is only one possible title.
- For the inductive step, assume n is an integer, $n \geq 1$, and that S_n is true. We want to see that S_{n+1} is true. So assume $b_1, b_2, \dots, b_n, b_{n+1}$ are $n + 1$ books. By the induction hypothesis, S_n is true, so $b_1, b_2, \dots, b_n$ all have the same title (call this title x). Since S_n is true (it works for any list of n books), $b_2, b_3, \dots, b_{n+1}$ also have the same title (call this title y). Since x and y are the title of the same book b_2, $x = y$. Thus $b_1, b_2, \dots, b_n, b_{n+1}$ all have the same title.

By the principle of mathematical induction, we deduce that all books have the same title.

(b) What is wrong with the following proof that the nth Fibonacci number is zero for all $n \geq 0$?

We prove by strong induction on the non-negative integer n, $n \geq 0$, the statement S_n: $F_n = 0$.

- For the base case, consider S_0: it says that $F_0 = 0$, and this holds by definition of F_0.

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- For the inductive step, assume $n \geq 1$ is a natural number and that S_m is true for all $m < n$. By definition, $F_n = F_{n-1} + F_{n-2}$. Since we are assuming S_m for all $m < n$, $F_{n-1} = 0$ and $F_{n-2} = 0$. Therefore $F_n = 0 + 0 = 0$. Thus S_n holds.

By the principle of mathematical induction, we deduce that $F_n = 0$ for all non-negative integers $n \geq 0$.

(5) The federal government issues two new bills, worth \$3 and \$7 respectively. Figure out all the amounts than can be paid using these bills. For example, one can pay \$10, since $10 = 7 + 3$, but one cannot pay \$8. *Hint: use strong induction.*

(6) Fix a natural number $n \geq 1$ and consider a $2^n \times 2^n$ chessboard with a corner removed. Prove that the board can be completely covered by non-overlapping trominoes (L-shaped pieces made out of three of the four squares of a 2×2 board).

(7) (Extra credit) Prove the following explicit formula for the nth Fibonacci number: $F_n = \frac{\phi^n - \psi^n}{\sqrt{5}}$, where $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$. *Hint: First show that $\phi^2 = \phi + 1$ and $\psi^2 = \psi + 1$.*