MATH 101 - SETS, GROUPS, AND TOPOLOGY, FALL 2018
ASSIGNMENT 5

Due Monday, September 24 at the beginning of class (please submit your assignment on Canvas). Make sure to include your full name *and the list of your collaborators* (if any) with your assignment. You may discuss problems with others, but you may *not* keep a written record of your discussions. Please refer to the syllabus for details.

In your proofs, you may assume basic facts about real numbers and integers, such as those mentioned on p. 90-91 of Hammack. You also can use results covered in class or results you have proven already. As a general rule, you should imagine that you are writing your proof to convince somebody else in the class who is very skeptical about the particular statement. In particular, it should be completely understandable to another student: always justify your reasoning in plain English.

(1) Write each of the following sets using set-builder notation:
  (a) $\{3, 9, 27, 81, \ldots\}$.
  (b) $\{0, \pi, 2\pi, 3\pi, 4\pi, \ldots\}$.
  (c) $\{\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, \sqrt{10}, \ldots\}$.
  (d) $\{-5, -3, -1, 1, 3, 5\}$.

(2) The *power set* of a set A (written $\mathcal{P}(A)$) is defined to be the set of all subsets of A (see section 1.4 in Hammack for more on the power set).
  (a) Write down the set $\mathcal{P}(\{4, 9, 13\})$ by listing its elements between braces.
  (b) Explain why $\emptyset \neq \mathcal{P}(\emptyset)$.
  (c) Write down the set $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset)))$ by listing all its elements between braces.
  (d) Assume A and B are sets. Show that $A \subseteq B$ if and only if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

(3) Assume A, B, and C are sets. Prove or disprove *(Note: to disprove, it is enough to give explicit examples of sets A, B, and C that fail to satisfy the statement)*:
  (a) $A - (B \cap C) = (A - B) \cup (A - C)$.
  (b) $A \cup (B - C) = (A \cup B) - (A \cup C)$.

(4) For a and b real numbers, we write (a, b) for the set $\{x \in \mathbb{R} \mid a < x < b\}$ and $[a, b]$ for $\{x \in \mathbb{R} \mid a \leq x \leq b\}$. We similarly define $(a, b]$ and $[a, b)$. Note that if $a > b$, then $[a, b] = \emptyset$, and if $a \geq b$, then $(a, b) = \emptyset$.
  (a) Give a very explicit description of $\bigcap_{b>0}(0, b)$, the intersection of all sets of the form $(0, b)$, for b a positive real number.
  (b) Give a very explicit description of $\bigcup_{a>0}[a, 1]$, the union of all sets of the form $[a, 1]$, for a a positive real number.

  Of course, you should in both cases *prove* that your description is correct.

(5) (Extra credit) In this problem, you will prove that the principle of mathematical induction is equivalent to the fact that any non-empty set of natural numbers has a minimal element.

Date: September 20, 2018.

First, a definition: for A a set of natural numbers, a number x is defined to be a *minimal element of* A if $x \in A$ and $x \leq y$ for any $y \in A$.

(a) Use induction to prove that any non-empty set of natural numbers has a minimal element. *Hint: show by strong induction on n that all sets of natural numbers containing n have a minimal element.*

(b) Conversely, prove that *if* any non-empty subset of natural numbers has a minimal element, *then* the principle of strong mathematical induction is true. *Hint: consider the set of natural numbers n for which S_n is false.*