MATH 101, FALL 2018: AXIOMS, FACTS, AND THEOREMS OF THE REAL LINE

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These notes give a list of *axioms*, *facts*, and *theorems* about numbers that we will see and use in this course. This is mostly meant as a quick reference to look up facts and theorems. We will call an *axiom* a statement that we just take for granted. A *fact* is a statement that follows from the axioms that we can use, but for which we skip the proof (usually because the proof is not too interesting). A *theorem* is a statement that follows from the axioms (and the facts) and that we will prove in this class.

Axioms of real numbers. There is:

- A set $\mathbb{R}$ (called the set of *real numbers*).
- Two binary operations $+$ and $\cdot$ on $\mathbb{R}$ (called *addition* and *multiplication*).
- A relation $<$ on $\mathbb{R}$ (called the *ordering* of $\mathbb{R}$).

with the following properties:

(A) $(\mathbb{R}, +)$ is an abelian group. We denote its identity element by 0, and write $-a$ for the inverse of an element a in $(\mathbb{R}, +)$.

(M) $(\mathbb{R} - \{0\}, \cdot)$ is an abelian group. Moreover, $\cdot$ is associative and commutative on all of $\mathbb{R}$ (including 0). We denote the identity element of $(\mathbb{R} - \{0\}, \cdot)$ by 1, and write a^{-1} for the inverse of an element a in $(\mathbb{R} - \{0\}, \cdot)$.

(D) Addition and multiplication satisfy the *distributive law*: for all $x, y, z \in \mathbb{R}$, $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$.

(O) The binary relation $<$ satisfies:
  - (O_1) Trichotomy: if $x \in \mathbb{R}$, then exactly one of the following is true: $x < 0$, $0 = x$, or $0 < x$.
  - (O_2) Sums and products of positives are positives: if $x, y \in \mathbb{R}$, $0 < x$ and $0 < y$, then $0 < x + y$ and $0 < x \cdot y$.
  - (O_3) Adding a fixed element preserves inequalities: if $x, y, z \in \mathbb{R}$ and $x < y$, then $x + z < y + z$.

(C) The completeness axiom: any non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound.

Some notation: when brackets are not present, multiplication should be done first, i.e. for x, y real numbers, $x \cdot y + x$ means $(x \cdot y) + x$, and *not* $x \cdot (y + x)$. We often write xy instead of $x \cdot y$. By associativity, the order of summation does not matter, so we write $x + y + z$ for $(x + y) + z$ (which is the same thing as $x + (y + z)$). Similarly for multiplication.

We write $y > x$ to mean $x < y$. We write $x \leq y$ to mean that $x < y$ or $x = y$. $x \geq y$ means $y \leq x$. We say x is *positive* if $0 < x$, *negative* if $x < 0$, *non-negative* if $0 \leq x$. When we want to emphasize that x is not zero, we may say "strictly positive" or "strictly negative".

Date: November 13, 2018.

Notice that it is necessary to explicitly define relations such as $>$ since all our axioms talk about is $<$. We can similarly define subtraction and division:

Definition 1. For real numbers x, y, we define $x - y$ to mean $x + (-y)$. Similarly, for y nonzero, we define x/y (also written $\frac{x}{y}$) to mean $x \cdot y^{-1}$.

From the axioms and the definitions of subtraction and division, we can go on to prove many more elementary properties. The arguments are usually quite boring (you will be asked to do a few of them in your homework). We list here all the elementary facts we will need (you can use them freely).

Fact 2 (Properties of addition and multiplication)**.** For all real numbers x, y, z, w:

- (F_0): $x \cdot 0 = 0$.
- (F_1): $-(xy) = (-x)y$.
- (F_2): $-x = (-1)x$.
- (F_3): $(-x)(-y) = xy$.
- (F_4): If $xy = 0$, then $x = 0$ or $y = 0$ (or both).
- (F_5):
  - $(x + y)(z + w) = xz + xw + yz + yw$,
  - $(x + y)^2 = x^2 + 2xy + y^2$.
  - $(x - y)^2 = x^2 - 2xy + y^2$.
  - $(x + y)(x - y) = x^2 - y^2$.

  (Note: as usual with groups, x^2 stands for $x \cdot x$))

Fact 3 (Properties of the ordering)**.** For all real numbers x, y, z, w:

- (F_6): Totality: Exactly one of $x < y$, $x = y$, $y < x$ always holds. Exactly one of $x \leq y$ or $y < x$ always holds.
- (F_7): Reflexivity: $x \leq x$.
- (F_8): Antisymmetry: If $x \leq y$ and $y \leq x$, then $x = y$.
- (F_9): Transitivity: If $x \leq y$ and $y \leq z$, then $x \leq z$. Similarly if $\leq$ is replaced by $<$.
- Interaction with addition and multiplication:
  - (F_{10}): $0 < 1$.
  - (F_{11}): If $x \leq y$ and $z \leq w$, then $x + z \leq y + w$. Similarly if $\leq$ is replaced by $<$.
  - (F_{12}): If $x \leq y$, then $-y \leq -x$. Similarly if $\leq$ is replaced by $<$.
  - (F_{13}): If $x \leq y$ and $0 \leq z$, then $xz \leq yz$.
  - (F_{14}): If $0 \leq x$ and $0 \leq y$, then $0 \leq xy$. Similarly if $\leq$ is replaced by $<$.
  - (F_{15}): $0 \leq x \cdot x$, and if $0 < x$ then $0 < x \cdot x$.
  - (F_{16}): If $0 < x$, then $0 < x^{-1}$.
  - (F_{17}): If $0 < x < y$, then $0 < y^{-1} < x^{-1}$.

Once we have the real numbers, we can also precisely define the integers, natural numbers, and rationals.

Definition 4. We define the following sets:

- The set $\mathbb{Z}$ (called the set of *integers*) is the subgroup of $(\mathbb{R}, +)$ generated by 1. That is, it is the intersection of all the subgroups of $(\mathbb{R}, +)$ which contain 1.
- The set $\mathbb{N}$ (called the set of *natural numbers*) is the set of strictly positive integers.

- The set $\mathbb{Q}$ (called the set of *rational numbers*) is defined to be the set $\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$.

The number 2 is defined to be $1+1$. Similarly, $3 = 1+1+1$, $4 = 1+1+1+1$, etc. Thus the *natural numbers* are is just the set containing the numbers $1, 2, 3, \ldots$, etc.

You may take the following facts about the natural numbers and integers for granted (but they actually follow from the axioms).

Fact 5.

(1) For all rationals x and y, $x + y$ and $x \cdot y$ are rationals.
(2) For all integers m and n, $m + n$ and $m \cdot n$ are integers.
(3) For all natural numbers m and n, $m + n$ and $m \cdot n$ are natural numbers.
(4) Any non-empty set S of natural numbers has a minimal element: an element $x \in S$ such that if $y \in S$, then $x \leq y$.
(5) The principle of mathematical induction holds.
(6) Divisions have remainders: for any integer n and any integer $d > 0$, there exists integers q and r (the quotient and remainder) such that $0 \leq r < d$ and $n = dq + r$.

We may use all these facts without explicitly mentioning them each time.

Square root and absolute value

Definition 6. A real number x is said to be a *square root* of a real number y if x is non-negative and $x^2 = y$.

By property (F_{15}) from Fact 2, x^2 is always non-negative, so *only non-negative real numbers have a real square root.* Moreover, the square root is unique:

Fact 7 (Uniqueness of the square root)**.** Given x, y non-negative real numbers, assume $x^2 = y^2$. Then $x = y$.

Using the completeness axiom, square roots exist:

Theorem 8 (1.4.5 in Abott)**.** Every non-negative real number has a square root.

Definition 9. For x a non-negative real number, we write $\sqrt{x}$ for the unique square root of x.

Warning. Assume that x, y are real numbers and $x^2 = y$. Do we have $x = \sqrt{y}$? *No*, because we do *not* know that x is non-negative. Indeed, it turns out that $-\sqrt{y}$ is also a possible solution, which will be different from $\sqrt{y}$ if $y > 0$. Using uniqueness of the square root, it is not hard to see that these are the only possible solutions.

How do square roots play with the ordering? It turns out taking a square root preserves the ordering.

Fact 10. For x, y real numbers, if $0 \leq x \leq y$, then $x^2 \leq xy \leq y^2$ and $\sqrt{x} \leq \sqrt{y}$.

Taking square root and squares preserve products:

Fact 11. For all real numbers x and y:

- $(xy)^2 = x^2y^2$.
- If x and y are non-negative, $\sqrt{xy} = \sqrt{x}\sqrt{y}$.

Definition 12. The *absolute value* $|x|$ of a real number x is defined by:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Fact 13 (Elementary properties of the absolute value)**.** For all real numbers x and y:

(1) $x^2 = |x|^2$.
(2) $|x| = \sqrt{x^2}$.
(3) $x \leq |x|$.
(4) $|xy| = |x||y|$.
(5) $|x| \geq 0$, and $|x| > 0$ if $x \neq 0$.
(6) $|-x| = |x|$.

Theorem 14 (The triangle inequality, 1.2.5 in Abbott)**.** For all real numbers x, y and z:

(1) $|x + y| \leq |x| + |y|$.
(2) $|x - y| \leq |x - z| + |z - y|$.

Theorem 15 (Proving that two numbers are equal, 1.2.6 in Abbott)**.** Assume x and y are real numbers. We have that $x = y$ if and only if $|x - y| < \epsilon$ for all $\epsilon > 0$.

The axiom of completeness

Fact 16. Every non-empty set of real numbers that is bounded below has a greatest lower bound.

Definition 17. For real numbers a and b, we define $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$, $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$, $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$, and $(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$. We call a set of the form (a, b) an *open interval* and a set of the form $[a, b]$ a *closed interval*. We also write (a, ∞) for the set $\{x \in \mathbb{R} \mid a < x\}$ and $(-\infty, a)$ for $\{x \in \mathbb{R} \mid x < a\}$. Similarly define $[a, \infty)$ and $(-\infty, a]$.

Note that ∞ and $-\infty$ are *not* members of $\mathbb{R}$: writing (a, ∞) is just notation. Also, do not confuse the open interval (a, b) with the ordered pair (a, b). Even though they are written in exactly the same way, they are completely different objects. Which is meant depends on context.

Theorem 18 (Nested interval property, 1.4.1 in Abbott)**.** Assume that for each natural number $n \in \mathbb{N}$, we are given a non-empty closed interval $[a_n, b_n]$. Assume $I_{n+1} \subseteq I_n$ for each $n \in \mathbb{N}$. Then $\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset$.

Theorem 19 (Archimedean property, 1.4.2 in Abbott)**.**

(1) Given any real number x, there exists a natural number $n \in \mathbb{N}$ such that $x < n$.
(2) Given any real number $y > 0$, there exists a natural number $n \in \mathbb{N}$ with $0 < \frac{1}{n} < y$.

Fact 20. Given any real number x, there exists a unique integer m such that $m \leq x < m + 1$ and a unique integer n such that $n < x \leq n + 1$.

Theorem 21 (The rationals are dense in the reals, 1.4.3 in Abbott)**.** For any real numbers a and b with $a < b$, there exists a rational number r with $a < r < b$.

Theorem 22 (Existence of the square root of two, 1.4.4 in Abbott)**.** There exists a real number α such that $\alpha^2 = 2$.

Fact 23 (Existence of nth root)**.** For any natural number n and any $y \geq 0$, there exists a unique $x \geq 0$ such that $x^n = y$. We write $\sqrt[n]{y}$ for this x.

Limits

Definition 24. A sequence (a_n) *converges* to a real number a if for every $\epsilon > 0$, there exists a natural number N such that whenever $n \geq N$, we have that $|a - a_n| < \epsilon$. A sequence that converges to a real number is called *convergent*. A sequence that does not converge is called *divergent* (and is said to *diverge*). We write $\lim(a_n) = a$, or $\lim_{n\to\infty}(a_n) = a$, or $(a_n) \to a$ to mean that the sequence (a_n) converges to a (the number a is called the *limit* of the sequence).

Fact 25 (Assignment 17)**.**
1. If (a_n) converges to a and (a_n) converges to b, then $a = b$ (that is, the limit is unique if it exists).
2. Assume c is a real number. If $a_n = c$ for all $n \in \mathbb{N}$, then $(a_n) \to c$.
3. If (a_n) and (b_n) are two sequences and their exists a natural number N such that $a_n = b_n$ for all $n \geq N$, then (a_n) is convergent if and only if (b_n) is convergent. Moreover, if they are convergent then (a_n) and (b_n) will have the same limit.
4. $(\frac{1}{n}) \to 0$.

Theorem 26 (Algebraic limit theorem (part I), 2.3.3(i) in Abbott)**.** If $(a_n) \to a$ and c is a real number, then $(ca_n) \to ca$.

Theorem 27 (Order limit theorem, 2.3.4 in Abbott)**.** If $(a_n) \to a$, $(b_n) \to b$, and $a_n \leq b_n$ for all n, then $a \leq b$.

Limits and subsequences

Definition 28. A sequence (a_n) is *bounded* if there exists a positive real number M such that $|a_n| \leq M$ for every natural number n.

Theorem 29 (2.3.2 in Abbott)**.** Any convergent sequence is bounded.

Definition 30. Given a sequence $(a_n)_{n\in\mathbb{N}}$ and natural numbers $n_1 < n_2 < n_3 < \ldots$, the sequence $(a_{n_k})_{k\in\mathbb{N}}$ is called a *subsequence* of (a_n).

Theorem 31 (2.5.2 in Abbott)**.** Any subsequence of a convergent sequence converges to the same limit as the original sequence.

Definition 32. A sequence (a_n) is:
- *increasing* if $a_n \leq a_m$ whenever $n \leq m$.
- *decreasing* if $a_n \geq a_m$ whenever $n \leq m$.
- *monotone* if increasing or decreasing.

Theorem 33 (Monotone convergence theorem, 2.4.2 in Abbott)**.** Any sequence which is monotone and bounded converges.

Theorem 34 (Bolzano-Weierstrass theorem, 2.5.5 in Abbott)**.** Any bounded sequence has a convergent subsequence.

Definition 35. A sequence (a_n) is *Cauchy* if for every $\epsilon > 0$ there exists a natural number N such that whenever $n, m \geq N$ we have $|a_n - a_m| < \epsilon$.

Theorem 36 (Assignment 17 and 2.6.4 in Abbott)**.** A sequence converges if and only if it is Cauchy.

Accumulation points of sequences

For the details of this section, see the supplementary notes posted on the course website.

Definition 37. An *accumulation point* (also called a *cluster point*) of a sequence (a_n) is a real number a such that some subsequence of (a_n) converges to a. We denote the set of accumulation points of the sequence (a_n) by $\text{acc}((a_n))$.

Theorem 38 (Basic properties of accumulation points)**.** Assume (a_n), (b_n) are sequences and a is a real number.

(1) If (a_{n_k}) is a subsequence of (a_n), then $\text{acc}((a_{n_k})) \subseteq \text{acc}((a_n))$.
(2) If $(a_n) \to a$, then $\text{acc}((a_n)) = \{a\}$.
(3) If (a_n) is bounded and $\text{acc}((a_n)) = \{a\}$, then $(a_n) \to a$.
(4) (a_n) has a bounded subsequence if and only if $\text{acc}((a_n)) \neq \emptyset$.
(5) If (a_n) is a bounded sequence, then $\text{acc}((a_n))$ is not empty, bounded below, and bounded above.
(6) If (a_n) and (b_n) are sequences such that for some natural number N, $a_n = b_n$ whenever $n \geq N$, then $\text{acc}((a_n)) = \text{acc}((b_n))$.

Definition 39 (Limit superior and inferior)**.** Given a bounded sequence (a_n), the *limit superior* of (a_n), written $\limsup a_n$, is defined to be the supremum of $\text{acc}((a_n))$. The *limit inferior* of (a_n), written $\liminf a_n$, is defined to be the infimum of $\text{acc}((a_n))$.

Theorem 40 (Basic properties of inf and sup)**.** Assume that A and B are nonempty sets of real numbers that are bounded below and bounded above.

(1) $\inf(A) \leq \sup(A)$.
(2) $\inf(A) = \sup(A)$ if and only if $A = \{a\}$ for some real number a (in this case, $a = \inf(A) = \sup(A)$).
(3) If for all $a \in A$, there exists $b \in B$ with $a \leq b$, then $\sup(A) \leq \sup(B)$.
(4) If for all $a \in A$ and all $b \in B$, $a \leq b$, then $\sup(A) \leq \inf(B)$.

Theorem 41 (Basic properties of limit superior and inferior)**.** Assume that (a_n) and (b_n) are bounded sequences.

(1) $\liminf a_n \leq \limsup a_n$.
(2) $\liminf a_n = \limsup a_n$ if and only if (a_n) is convergent. In this case, $\lim a_n = \liminf a_n = \limsup a_n$.
(3) If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $\liminf a_n \leq \liminf b_n$ and $\limsup a_n \leq \limsup b_n$.

Theorem 42 (Squeeze theorem)**.** Assume that (a_n), (b_n), (c_n) are sequences with $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$. If $(a_n) \to \ell$ and $(c_n) \to \ell$, then $(b_n) \to \ell$.

Theorem 43 (Strengthened squeeze theorem, assignment 18)**.** Assume that (a_n), (b_n), (c_n) are sequences. Assume that $(a_n) \to \ell$ and $(c_n) \to \ell$. Assume further that for all $\epsilon > 0$ there exists a natural number N such that whenever $n \geq N$, we have that $a_n - \epsilon \leq b_n \leq c_n + \epsilon$. Then $(b_n) \to \ell$.

Theorem 44 (Algebraic limit theorem, part II)**.** Assume $(a_n), (b_n)$ are sequences and a, b are real numbers. Then:

(1) If $(a_n) \to a$, then $(a_n + b) \to a + b$.
(2) $(a_n) \to 0$ if and only if $(|a_n|) \to 0$.
(3) $(a_n) \to a$ if and only if $(|a_n - a|) \to 0$.
(4) If $(b_n) \to b$ and $b \neq 0$, then there exists a real number $\delta > 0$ and a natural number N such that $|b_n| \geq \delta$ whenever $n \geq N$.
(5) If $(a_n) \to a$ and $(b_n) \to b$, then $(a_n + b_n) \to a + b$.
(6) If $(a_n) \to a$ and $(b_n) \to b$, then $(a_n b_n) \to ab$.
(7) If $(a_n) \to a$, $(b_n) \to b$, $b \neq 0$, and $b_n \neq 0$ for all $n \in \mathbb{N}$, then $(\frac{a_n}{b_n}) \to \frac{a}{b}$.