MATH 101, FALL 2018: SUPPLEMENTARY NOTES ON THE REAL LINE

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These notes describe the material for November 26, 2018 (while similar content is in Abbott's book, the presentation here is different).

Before Thanksgiving break, we saw that to check whether a sequence is convergent using the definition of convergence, one must have a value for the limit a. This can be annoying if we do not know the limit yet, but still want to prove on general grounds that it exists. Another technical problem was how to prove, easily, that a sequence is divergent (i.e. does not converge).

To that end, we first observed that any convergent sequence is bounded, so (by the *contrapositive*), a sequence that is *not* bounded is divergent. We also saw that any subsequence of a convergent sequence converges to the same limit as the original sequence. Thus if we have a sequence which has two convergent subsequences with different limits, we can directly see that the original sequence is divergent.

For example the sequence $(0, 1, 0, 1, 0, 1, \ldots)$ is divergent as it has a constantly zero subsequence and a constantly 1 subsequence. Unfortunately, the same sequence show that it is not true that a bounded sequence must always be convergent. However we saw two approximations: the *monotone convergence theorem*, which says that a monotone bounded sequence converges, and the *Bolzano-Weierstrass theorem*, which says that a bounded sequence always has a convergent subsequence. We used the latter to study Cauchy sequences and give a general criteria for convergence of a sequence (without using the limit).

It still would be nice to be able to define some kind of limit for any bounded sequence. Now unfortunately a bounded sequence may have several convergent subsequences, and they may not all converge to the same limit. For example as said before the sequence $(0, 1, 0, 1, \ldots)$ has the constantly zero subsequence and the constantly 1 subsequence. However we can look at the *set* of all such limits:

Definition 1. An *accumulation point* (also called a *cluster point*) of a sequence (a_n) is a real number a such that some subsequence of (a_n) converges to a. We denote the set of accumulation points of the sequence (a_n) by $\operatorname{acc}((a_n))$.

The reason for the name "accumulation point" is that if a is an accumulation point of (a_n), then there will be infinitely-many terms of (a_n) which will "accumulate" very close from a.

So for example $\operatorname{acc}((0, 1, 0, 1, 0, 1, 0, 1, \ldots)) = \{0, 1\}$ (try to prove this precisely as an exercise!). Let us study some general basic properties of accumulation points:

Theorem 2 (Basic properties of accumulation points)**.** Assume (a_n), (b_n) are sequences and a is a real number.

(1) If (a_{n_k}) is a subsequence of (a_n), then $\operatorname{acc}((a_{n_k})) \subseteq \operatorname{acc}((a_n))$.

Date: November 13, 2018.

(2) If $(a_n) \to a$, then $\mathrm{acc}((a_n)) = \{a\}$.
(3) If (a_n) is bounded and $\mathrm{acc}((a_n)) = \{a\}$, then $(a_n) \to a$.
(4) (a_n) has a bounded subsequence if and only if $\mathrm{acc}((a_n)) \neq \emptyset$.
(5) If (a_n) is a bounded sequence, then $\mathrm{acc}((a_n))$ is not empty, bounded below, and bounded above.
(6) If (a_n) and (b_n) are sequences such that for some natural number N, $a_n = b_n$ whenever $n \geq N$, then $\mathrm{acc}((a_n)) = \mathrm{acc}((b_n))$.

Proof.

(1) Assume that x is an accumulation point of (a_{n_k}). Then because a subsequence of (a_{n_k}) is a subsequence of the original sequence (a_n), x is an accumulation point of (a_n) as well.
(2) Assume that $(a_n) \to a$. Then by 2.5.2 in Abbott, any subsequence of (a_n) also converges to a, so $\mathrm{acc}((a_n)) = \{a\}$.
(3) This will be a problem on assignment 18.
(4) Assume that (a_n) has a bounded subsequence (a_{n_k}). Then (a_{n_k}) has a convergent subsequence by the Bolzano-Weierstrass theorem, so $\mathrm{acc}((a_{n_k})) \neq \emptyset$. By the first part, also $\mathrm{acc}((a_n)) \neq \emptyset$. Conversely, assume that $\mathrm{acc}((a_n)) \neq \emptyset$. Then (a_n) has an accumulation point, hence in particular a convergent subsequence (a_{n_k}). By 2.3.2 in Abbott, any convergent sequence is bounded, so (a_{n_k}) is a bounded subsequence of (a_n).
(5) This will be a problem on assignment 18.
(6) Because, essentially, being a convergent subsequence does not depend on the first few terms of a sequence (the details are left to you as an exercise).

$\square$

Note that by the part (5) of Theorem 2 and the axiom of completeness, we can take the inf and sup of the set of accumulation points of any bounded sequence. This will be a replacement for taking the "limit" of any bounded sequence. We give these numbers a name:

Definition 3 (Limit superior and inferior)**.** Given a bounded sequence (a_n), the *limit superior* of (a_n), written $\limsup a_n$, is defined to be the supremum of $\mathrm{acc}((a_n))$. The *limit inferior* of (a_n), written $\liminf a_n$, is defined to be the infimum of $\mathrm{acc}((a_n))$.

Before studying the basic properties of lim inf and lim sup, it is worth recalling a few simple properties of inf and sup:

Theorem 4 (Basic properties of inf and sup)**.** Assume that A and B are non-empty sets of real numbers that are bounded below and bounded above.

(1) $\inf(A) \leq \sup(A)$.
(2) $\inf(A) = \sup(A)$ if and only if $A = \{a\}$ for some real number a (in this case, $a = \inf(A) = \sup(A)$).
(3) If for all $a \in A$, there exists $b \in B$ with $a \leq b$, then $\sup(A) \leq \sup(B)$.
(4) If for all $a \in A$ and all $b \in B$, $a \leq b$, then $\sup(A) \leq \inf(B)$.

Proof. Assignment 18. $\square$

Theorem 5 (Basic properties of limit superior and inferior)**.** Assume that (a_n) and (b_n) are bounded sequences.

(1) $\liminf a_n \leq \limsup a_n$.
(2) $\liminf a_n = \limsup a_n$ if and only if (a_n) is convergent. In this case, $\lim a_n = \liminf a_n = \limsup a_n$.
(3) If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $\liminf a_n \leq \liminf b_n$ and $\limsup a_n \leq \limsup b_n$.

Proof. Let $A = \mathrm{acc}((a_n))$ and let $B = \mathrm{acc}((b_n))$.

(1) We have that $\liminf a_n = \inf(A) \leq \sup(A) = \limsup(a_n)$, where the equalities are by definition of lim inf and lim sup, and the inequality by Theorem 4(1).
(2) Assume that $\liminf a_n = \limsup a_n$. Then $\sup(A) = \inf(A)$, so by Theorem 4(2), $A = \{a\}$ for some real number a, and $a = \inf(A) = \sup(A)$. By Theorem 2(3), $(a_n) \to a$.

   Conversely, if $(a_n) \to a$, then by Theorem 2(2), $A = \{a\}$, so $\liminf(a_n) = \inf(A) = \sup(A) = \limsup(a_n)$ by Theorem 4(2).
(3) Assume that $a_n \leq b_n$ for all $n \in \mathbb{N}$. We show that $\limsup a_n \leq \limsup b_n$, and the proof for lim inf will be similar. We show that for all $a \in A$ there exists $b \in B$ such that $a \leq b$. This will be enough by Theorem 4(3). So assume $a \in A$. By definition of an accumulation point, there exists a subsequence (a_{n_k}) of (a_n) converging to a. Consider the corresponding subsequence (b_{n_k}). It may not be convergent, but by Bolzano-Weierstrass we can take a convergent subsequence $(b_{n_{k_m}})$ of it. Say this converges to b. By 2.5.2 in Abbott, the corresponding subsequence $(a_{n_{k_m}})$ of (a_n) converges to a. Since we know that $a_{n_{k_m}} \leq b_{n_{k_m}}$ for all m, the order limit theorem implies that $a \leq b$.

□

Limit inferior and superior are a useful tool to prove the following result:

Theorem 6 (Squeeze theorem)**.** Assume that (a_n), (b_n), (c_n) are sequences with $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$. If $(a_n) \to \ell$ and $(c_n) \to \ell$, then $(b_n) \to \ell$.

Proof. We compute the limit inferior and superior of (b_n) and show that they coincide. We have:

$$\ell = \lim a_n = \liminf a_n \leq \liminf b_n \leq \liminf c_n = \lim c_n = \ell$$

Where we have used that $\liminf a_n = \lim a_n$, because (a_n) is convergent (Theorem 5(2)). Therefore $\liminf b_n = \ell$. Similarly, $\limsup b_n = \ell$. It follows from Theorem 5(2) that (b_n) is convergent and $\lim b_n = \ell$. □

The following is a slightly stronger version that will also be useful:

Theorem 7 (Strengthened squeeze theorem, assignment 18)**.** Assume that (a_n), (b_n), (c_n) are sequences. Assume that $(a_n) \to \ell$ and $(c_n) \to \ell$. Assume further that for all $\epsilon > 0$ there exists a natural number N such that whenever $n \geq N$, we have that $a_n - \epsilon \leq b_n \leq c_n + \epsilon$. Then $(b_n) \to \ell$.

We can use the strengthened squeeze theorem to prove the rest of the algebraic limit theorem:

Theorem 8 (Algebraic limit theorem, part II)**.** Assume $(a_n), (b_n)$ are sequences and a, b are real numbers. Then:

(1) If $(a_n) \to a$, then $(a_n + b) \to a + b$.
(2) $(a_n) \to 0$ if and only if $(|a_n|) \to 0$.
(3) $(a_n) \to a$ if and only if $(|a_n - a|) \to 0$.
(4) If $(b_n) \to b$ and $b \neq 0$, then there exists a real number $\delta > 0$ and a natural number N such that $|b_n| \geq \delta$ whenever $n \geq N$.
(5) If $(a_n) \to a$ and $(b_n) \to b$, then $(a_n + b_n) \to a + b$.
(6) If $(a_n) \to a$ and $(b_n) \to b$, then $(a_n b_n) \to ab$.
(7) If $(a_n) \to a$, $(b_n) \to b$, $b \neq 0$, and $b_n \neq 0$ for all $n \in \mathbb{N}$, then $(\frac{a_n}{b_n}) \to \frac{a}{b}$.

Proof.

(1) Assignment 18.
(2) Assignment 18.
(3) Assignment 18.
(4) Assignment 18.
(5) We first claim that for every $\epsilon > 0$, there exists a natural number N such that whenever $n \geq N$,

$$a_n + b - \epsilon \leq a_n + b_n \leq a_n + b + \epsilon$$

Indeed, assume $\epsilon > 0$. Since $(b_n) \to b$, there exists N such that $|b_n - b| < \epsilon$ whenever $n \geq N$. For such n, we can write equivalently that $b - \epsilon < b_n < b + \epsilon$. Adding a_n to this inequality, we get the claim.

Now we can use the strengthened squeeze theorem because we have that $(a_n + b) \to a + b$ (using the first part).

(6) By the third part, it suffices to show that $(|a_n b_n - ab|) \to 0$. For any natural number n, the triangle inequality gives:

$$|a_n b_n - ab| \leq |a_n b_n - a_n b| + |a_n b - ab|$$

Let's estimate each term. We have that $|a_n b_n - a_n b| = |a_n(b_n - b)|$. We know that (a_n) is convergent, so it is bounded: let M be a positive real number such that $|a_n| \leq M$ for all n. Then $|a_n(b_n - b)| \leq M|b_n - b|$. We have shown that:

$$0 \leq |a_n b_n - a_n b| \leq M|b_n - b|$$

The left hand side of this inequality (seen as the constantly zero sequence) goes to zero. The right hand, the sequence $(M|b_n - b|)$ also goes to zero because $(b_n) \to b$, so by previously proven parts $(|b_n - b|) \to 0$ and hence (algebraic limit theorem, part I), $(M|b_n - b|) \to M \cdot 0 = 0$. By the squeeze theorem, $(|a_n b_n - a_n b|) \to 0$.

Now let's estimate $|a_n b - ab| = |(a_n - a)b|$. Since $(a_n) \to a$, a similar argument gives that $|(a_n - a)b| \to 0$.

Putting all this together, we get (using the previous part) that $(|a_n b_n - a_n b| + |a_n b - ab|) \to 0$. By the squeeze theorem (using the constantly zero sequence on the left hand side), $|a_n b_n - ab| \to 0$.

(7) We have that $\frac{a_n}{b_n} = a_n \cdot \frac{1}{b_n}$, so if we can show that $(\frac{1}{b_n}) \to \frac{1}{b}$, we will be able to use the previous part. As before, it suffices to show that $(|\frac{1}{b_n} - \frac{1}{b}|) \to 0$. We have that $\frac{1}{b_n} - \frac{1}{b} = \frac{b - b_n}{b_n b}$. The numerator will go to zero, but we need to bound the denominator from *below*. This is given by the fourth part of the theorem: there exists a real number $\delta > 0$ and a natural number N such

that for any natural number $n \geq N$, $|b_n| \geq \delta$. Since convergence does not depend on finitely-many terms (assignment 17), we might as well assume that $N = 1$, i.e. that $|b_n| \geq \delta$ for *all* natural numbers n. Then:

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{|b - b_n|}{|b_n b|} \leq \frac{1}{\delta |b|}|b - b_n|$$

The right hand side goes to zero, so by the squeeze theorem, the left hand side must also go to zero, as desired.

$\square$