# MATH 123 - ALGEBRA II - SPRING 2020
# ASSIGNMENT 1

**Due Tuesday, February 4, 11h59pm.** (please submit your assignment as a PDF on Canvas). Unless otherwise noted, references are to Dummit and Foote, *Algebra*, 3rd edition, Wiley, 2004.

In this problem set, "ring" means "ring with identity".

## Practice problems

(*Not* for credit. No need to submit your solutions. Try to do them while minimally looking at the textbook)

(1) Show that the commutativity of addition in a ring follows from the other axioms.
(2) Show that in a ring, $0 \cdot a = 0$ for any $a$.
(3) Let $R$ be a commutative ring where $0 \neq 1$. Show that $R$ is an integral domain if and only if for any nonzero $a$ and any $b, c \in R$, $ab = ac$ implies $b = c$.
(4) Prove that, in $\mathbb{Z}/n\mathbb{Z}$, a nonzero element is a zero divisor if and only if it is not coprime to $n$.
(5) Prove that a finite integral domain is a field.
(6) True or false?
  (a) Any integral domain is a field.
  (b) Any field is an integral domain.
  (c) Any division ring is a field.
  (d) Any division ring is an integral domain.
  (e) The zero ring is a field.
  (f) In any ring, $(ab)^2 = a^2b^2$.
(7) DF, 7.1.5.
(8) (DF, 7.1.21) Let $X$ be any set, and for $A, B \in \mathcal{P}(X)$, define $A \cdot B$ to be $A \cap B$ and $A + B$ to be $(A - B) \cup (B - A)$ (the *symmetric difference* of $A$ and $B$, also denoted $A \Delta B$). Prove that under these operations $\mathcal{P}(X)$ is a ring, and moreover is Boolean: $A^2 = A$ for any $A$.

## Problems for credit

(1) (Extra credit: 15%)
  (a) Please fill in the survey at: `http://math.harvard.edu/~sebv/123-spring-2020/questionnaire.odt`. Submit it separately on Canvas.
  (b) I like to know my students as human beings, so I would like to have a short one on one 5-10 min chat with you during the first few weeks of the semester, just so that I can know your face, name, and a little bit about your background. Don't be afraid, we're not going to talk math (unless you really want to!). You don't need to prepare anything for the meeting.
  Please send me a short email at `sebv@math.harvard.edu` with subject "123 short meeting" and ask e.g. "is 1pm next Monday okay?". I will either reply yes or propose another time. The meeting will take place in my office, SC 321H.
(2) (DF, 7.1.1) Prove from the axioms that $(-1)(-1) = 1$ in a ring.
(3) (DF, 7.1.11) Prove that if $R$ is an integral domain and $x^2 = 1$, then $x = 1$ or $x = -1$.
(4) (DF, 7.1.15) A *Boolean ring* is a ring where $a^2 = a$ for any $a$. Prove that any Boolean ring is commutative. *Hint: consider* $(a + b)^2$.

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*Date*: January 24, 2020.

(5) Let $X$ be a set and $+, \cdot$ be two binary operations on $X$ such that both have identities, and for any $a, b, c, d \in X$:

$$(a + b) \cdot (c + d) = (a \cdot c) + (b \cdot d)$$

Prove that $+$ and $\cdot$ are the same operation, and that this operation is commutative.

(6) A *subring* of a ring $R$ is a subset of $R$ that contains 0, 1 and is closed under addition and multiplication. For example, $\mathbb{Q}$ is a subring of $\mathbb{R}$, but $\{0, 1\}$ is not a subring of $\mathbb{R}$.
   (a) (DF, 7.1.3) Assume $S$ is a subring of $R$. Show that if $u$ is a unit in $S$, then it is a unit in $R$. Give an example showing the converse is false.
   (b) (DF, 7.1.12) Show that any subring of a field is an integral domain.
   (c) (DF, 7.1.7) The *center* of a ring $R$ is the set $\{z \in R \mid zr = rz \text{ for all r } \in R\}$. Prove that the center is a subring of $R$.
   (d) (DF, 7.1.8) Describe the center of the quarternions (in other words, give a simple description of it in terms of well-known rings).