MATH 123 - ALGEBRA II - SPRING 2020
ASSIGNMENT 2

Due Tuesday, February 11, 11h59pm. (please submit your assignment as a PDF on Canvas). Unless otherwise noted, references are to Dummit and Foote, *Algebra*, 3rd edition, Wiley, 2004.

Remember that in this class "ring" means "ring with identity".

Practice problems

(*Not* for credit. No need to submit your solutions. Try to do them while minimally looking at the textbook)

(1) Give the definition of the following: zero ring, field, division ring, integral domain, unit, $R^\times$, ideal, homomorphism, $R[x]$, principal ideal.
(2) State and prove the first isomorphism theorem for rings.
(3) (DF, 7.3.3) Find all homomorphic images of $\mathbb{Z}$.
(4) Prove that an ideal I in a commutative ring R is maximal if and only if R/I is a field.
(5) Prove that any nonzero homomorphism with domain a field is injective.
(6) True or false?
   (a) Every ring has maximal ideals.
   (b) There is a unique homomorphism from the zero ring into any other ring.
   (c) There is a unique homomorphism from any ring into the zero ring.
   (d) If R is a ring and $a \in R$, then $(a) = \{ra \mid r \in R\}$.
   (e) In $\mathbb{Z}[x]$, every ideal is principal.

Problems for credit

(1) You should have the assignment from another student available for review in your Canvas todo list. Review problem 5 from that assignment. Refer to the peer review instructions on the course website for more details. *You are encouraged not to look at the official solution before submitting your review!*
(2) Let R be a ring. A *congruence relation* on R is an equivalence relation $\sim$ such that for any $a, a', b, b' \in R$, if $a \sim b$ and $a' \sim b'$ then $a + a' \sim b + b'$ and $a \cdot a' \sim b \cdot b'$.
   (a) Let I be an ideal of R. Prove that the relation $\sim$ defined by $a \sim b$ if and only if $a - b \in I$ is a congruence relation. How is the quotient R/I related to $\sim$?
   (b) Conversely, show that if $\sim$ is a congruence relation, then the equivalence class of 0 is an ideal.
   (c) Define an appropriate notion of congruence relation for groups, and prove analogously to the above that:
      (i) Any normal subgroup naturally yields a congruence relation.
      (ii) Any congruence relation naturally yields a normal subgroup.
(3) The ring $\mathbb{R}[x]/(x^2 + 1)$ is isomorphic to a well known ring: which one, and why?
(4) (DF, 7.3.29) Let R be a commutative ring. An element a of R is called *nilpotent* if $a^n = 0$ for some integer $n \geq 0$. Prove that the set of nilpotent elements forms an ideal.
(5) Let R be a commutative ring. If R is an integral domain, is $R[x]$ an integral domain? What about the converse?
(6) (DF, 7.4.8) Let R be an integral domain, and let $a, b \in R$. Prove that $(a) = (b)$ if and only if $a = ub$ for some unit u.

Date: February 5, 2020.

(7) (DF, 7.4.16) Consider $x^4 - 16$ as an element of the polynomial ring $E = \mathbb{Z}[x]$.
   (a) Find a polynomial of degree at most 3 which in $E/(x^4 - 16)$ is equal to $7x^{13} - 11x^9 + 5x^5 - 2x^3 + 3$.
   (b) Prove that, in $E/(x^4 - 16)$, $x - 2$ and $x + 2$ are zero divisors.

(8) (DF, 7.4.23) Prove that in a Boolean ring, every prime ideal is maximal.