MATH 123 - ALGEBRA II - SPRING 2020
ASSIGNMENT 3

Due Tuesday, February 18, 11h59pm. (please submit your assignment as a PDF on Canvas). Unless otherwise noted, references are to Dummit and Foote, *Algebra*, 3rd edition, Wiley, 2004.

Remember that in this class "ring" means "ring with identity".

Practice problems

(*Not* for credit. No need to submit your solutions. Try to do them while minimally looking at the textbook)

(1) Give the definition of the following: maximal ideal, prime ideal, direct product of two rings, $I + J$, $I \cdot J$, field of fractions, Euclidean domain, unique factorization domain, irreducible, prime, associates.
(2) State and prove the Chinese remainder theorem for commutative rings.
(3) Prove that in a commutative ring, an ideal is prime if and only if the quotient is an integral domain.
(4) Prove that every Euclidean domain is a principal ideal domain.
(5) Prove that every principal ideal domain is a unique factorization domain.
(6) True or false?
   (a) In an integral domain, every irreducible element is prime.
   (b) In a principal ideal domain, every prime element is irreducible.
   (c) Being associate is an equivalence relation.
   (d) In $\mathbb{Z}[\sqrt{-5}]$, 2 is prime.
   (e) In $\mathbb{Q}[x]$, the only associates of x^2 are x^2 and $-x^2$.

Problems for credit

(1) You should have the assignment from another student available for review in your Canvas todo list. Review problem 3 from that assignment (submit your comment on Canvas, *not* with this assignment). Refer to the peer review instructions on the course website for more details. *You are encouraged not to look at the official solution before submitting your review!*
(2) (DF, 7.4.37) A commutative ring R is called a *local ring* if it has a unique maximal ideal.
   (a) Prove that if R is a local ring with maximal ideal M, then every element of $R - M$ is a unit.
   (b) Conversely, prove that if R is a commutative ring in which the set of nonunits forms an ideal M, then R is a local ring with unique maximal ideal M.
(3) (DF, 7.5.3) Let F be a field. Prove that F contains a unique smallest subfield F_0, which is isomorphic to either $\mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$ for some prime p.
(4) Let R and S be rings.
   (a) (DF, 7.6.3) Prove that any ideal of $R \times S$ is of the form $I \times J$, for I an ideal of R and J an ideal of S.
   (b) (DF, 7.6.4) Show that if R and S are not zero rings, $R \times S$ is not a field.
(5) (DF, 8.2.3) Show that the quotient of a PID by a prime ideal is again a PID. Is it still true that the quotient of a PID by an arbitrary ideal is a PID?
(6) (DF, 8.2.7) An integral domain R in which every ideal generated by two elements is principal is called a *Bezout domain*.

Date: February 11, 2020.

(a) Prove that the integral domain R is a Bezout domain if and only if every pair of elements a, b of R has a greatest common divisor[1] that can be written as $ax + by$ for some $x, y \in R$.

(b) Let F be the field of fractions of a Bezout domain R. Prove that every element of F can be written in the form $\frac{a}{b}$, with $a, b \in R$ and a, b relatively prime (that is, with 1 as a greatest common divisor).

(7) (DF, 8.3.11) Prove that a ring R is a PID if and only if it is a UFD which is also a Bezout domain.

[1]A *greatest common divisor* of a and b is an element d such that $(a, b) \subseteq (d)$ and for any principal ideal I containing (a, b), $(d) \subseteq I$.