MATH 123 - ALGEBRA II - SPRING 2020
ASSIGNMENT 5

Due Tuesday, March 3, 11h59pm. (please submit your assignment as a PDF on Canvas). Unless otherwise noted, references are to Dummit and Foote, *Algebra*, 3rd edition, Wiley, 2004.

Remember that in this class "ring" means "ring with identity". Everywhere below, R denotes a ring.

Practice problems

(*Not* for credit. No need to submit your solutions. Try to do them while minimally looking at the textbook)

(1) Give the definition of the following: external direct sum, internal direct sum, free module, basis of a module, rank of a module, Noetherian, elementary divisors, invariant factors, Jordan canonical form.
(2) State and prove the structure theorem for modules seen in class (both forms).
(3) For what kind of integral domains is a submodule of a free module free, and why?
(4) Prove that any square matrix of complex numbers has a Jordan canonical form.
(5) True or false?
    (a) A module is Noetherian if and only if it is finitely generated.
    (b) Any abelian group is isomorphic to a product of cyclic groups.
    (c) For finitely generated modules over PIDs, free is equivalent to torsion-free.
    (d) For R an integral domain, any free R-module of rank n is isomorphic to R^n.
    (e) For R an integral domain, any submodule of a finitely generated free module is finitely generated.

Problems for credit

(1) You should have the assignment from another student available for review in your Canvas todo list. Review problem 3 from that assignment (submit your comment on Canvas, *not* with this assignment). Refer to the peer review instructions on the course website for more details. *You are encouraged not to look at the official solution before submitting your review!*
(2) (a) Let M_1, M_2 be R-modules and N_1, N_2 be submodules of M_1 and M_2 respectively. Show that $(M_1 \times M_2)/(N_1 \times N_2) \cong (M_1/N_1) \times (M_2/N_2)$.
    (b) Let R be an integral domain and let M be an R-module. Let $y \in M$ and $a \in R$. Assume that y is not a torsion element. Show that $(Ry)/(Ray) \cong R/(a)$. Give an example showing that this may no longer hold if y is a torsion element.
    (c) Assume R is a commutative ring with $0 \neq 1$. Show that if $R^n \cong R^m$ (as R-modules), then $n = m$. *Hint: take a maximal ideal of R, and quotient by a suitable submodule. Be careful: we are NOT asuming that R is an integral domain here, so you cannot use the proposition about the rank proven in class.*
(3) (a) (DF, 10.1.9) If N is a submodule of M, recall that the *annihilator of N in R*, $\text{Ann}(N)$, is defined to be $\{r \in R \mid rn = 0 \text{ for all } n \in N\}$. Show that the annihilator of N in R is an ideal of R.

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   (b) (DF, 10.1.10) If I is an ideal of R, the *annihilator of I in M* is defined to be $\{m \in M \mid rm = 0 \text{ for all } r \in I\}$. Prove that the annihilator of I in M is a submodule of M.
   (c) (DF, 10.1.11) Let M be the abelian group $\mathbb{Z}/24\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z} \times \mathbb{Z}/50\mathbb{Z}$, considered as a $\mathbb{Z}$-module.
      (i) Give a generator for the annihilator of M in $\mathbb{Z}$.
      (ii) Describe the annihilator of $2\mathbb{Z}$ in M as a product of cyclic groups.

(4) Assume that R is commutative with $0 \neq 1$.
   (a) Prove that all R-modules are free if and only if R is a field.
   (b) (DF, 10.3.13) Let F be a free R-module of finite rank. Prove that $\mathrm{Hom}_R(F, R) \cong F$. Here, $\mathrm{Hom}_R(F, R)$ is the R-module of all homomorphisms from F to R, under the natural action and addition.

(5) (a) Describe all abelian groups of order 72.
   (b) For each of the following abelian groups, determine whether it is finitely generated. If it is, give its free rank, elementary divisors, and invariant factors. We write $\mathbb{Z}_n$ for $\mathbb{Z}/n\mathbb{Z}$.
      (i) $\mathbb{Z} \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{16} \oplus \mathbb{Z}$.
      (ii) $\mathbb{Q}$, with addition.
      (iii) $\mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3$.

(6) (DF, 12.1.8) Let R be a PID, let M be a (not necessarily finitely generated!) torsion R-module, and let p be a prime in R. Show that if there exists a nonzero $m \in M$ such that $pm = 0$, then $\mathrm{Ann}(M) \subseteq (p)$.

(7) (DF, 12.1.15) Show that if R is a Noetherian ring, then R^n is a Noetherian R-module.