MATH 123 - ALGEBRA II - SPRING 2020
ASSIGNMENT 9

Due Tuesday, April 7, 11h59pm. (please submit your assignment as a PDF on Canvas). Unless otherwise noted, references are to Dummit and Foote, *Algebra*, 3rd edition, Wiley, 2004.

Practice problems

(*Not* for credit. No need to submit your solutions. Try to do them while minimally looking at the textbook)

(1) Give the definition of the following: Frobenius map, perfect field, separable and inseparable degree, cyclotomic polynomial, automorphism, fixed field, Galois extension, Galois group.
(2) What is the degree of the cyclotomic field of nth root of unity? Prove that your answer is correct.
(3) Prove that in a perfect field, irreducible polynomials are separable.
(4) Prove that if K/F is a finite extension, then $|\operatorname{Aut}(K/F)| \leq [K : F]$. Give an example where equality holds, and an example where the inequality is strict.
(5) Compute the Galois group of $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$.
(6) True or false?
  (a) If f is a polynomial over a field of characteristic p with zero derivative, then $f(x) = g(x^p)$ for some polynomial g.
  (b) $\mathbb{Q}(e^{2\pi i/n})$ has degree $n - 1$ over $\mathbb{Q}$.
  (c) p is prime if and only if the pth cyclotomic polynomial is of the form $x^{p-1} + x^{p-2} + \ldots + x + 1$.
  (d) If K/F is Galois, and L/K is Galois, then L/F is Galois.

Problems for credit

(1) You should have the assignment from another student available for review in your Canvas todo list. Review problem 3 from that assignment (submit your comment on Canvas, *not* with this assignment). Refer to the peer review instructions on the course website for more details. *You are encouraged not to look at the official solution before submitting your review!*
(2) Let F be a field and let $f(x) \in F[x]$ be a polynomial of degree $n \geq 1$. Assume that the characteristic of F is either zero or at least n. Let $\alpha \in F$. Let $1 \leq m \leq n$. Show that α is a root of f of multiplicity at least m if and only if $f^{(k)}(\alpha) = 0$ for all $k \in \{0, 1, 2, \ldots, m - 1\}$.

   Here, $f^{(k)}$ denotes the kth derivative of f: $f^{(0)} = f$, and $f^{(k+1)}$ is the derivative of $f^{(k)}$.
(3) (a) Show that any algebraically closed field is perfect.
  (b) (DF, 13.6.4) Prove that if $n = p^k m$ where p is prime and m is coprime to p, then there are precisely m distinct nth root of unity over a field of characteristic p.
  (c) Show that a field F is perfect if and only if any irreducible polynomial in $F[x]$ is separable.
(4) (a) (DF, 14.1.2) Prove that the map $\tau : \mathbb{C} \to \mathbb{C}$ defined by $\tau(a+bi) = a-bi$ (called *complex conjugation*) is an automorphism of $\mathbb{C}$.
  (b) (DF, 14.1.3) Determine the fixed field of complex conjugation on $\mathbb{C}$.
  (c) (DF, 14.1.4) Prove that $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ are not isomorphic.
(5) Determine $\operatorname{Aut}(\mathbb{R}/\mathbb{Q})$. *Hint: first show that any automorphism must preserve the ordering on $\mathbb{R}$ and be continuous.*

Date: April 6, 2020.