MATH 123 - ALGEBRA II - SPRING 2020
ASSIGNMENT 11

Due Friday, April 24, 11h59pm. (please submit your assignment as a PDF on Canvas). Unless otherwise noted, references are to Dummit and Foote, *Algebra*, 3rd edition, Wiley, 2004.

Practice problems

(*Not* for credit. No need to submit your solutions. Try to do them while minimally looking at the textbook)

(1) Give the definition of the Galois closure, and of the notion of a separable extension.
(2) Prove that if K/F is a Galois extension and E is an intermediate field, then E/F is Galois if and only if $\mathrm{Aut}(K/E)$ is normal in $\mathrm{Aut}(K/F)$.
(3) State and prove the primitive element theorem.
(4) State and prove the characterization of the numbers n such that the regular n-gon can be constructed with straightedge and compass.
(5) True or false?
   (a) Any finite extension of $\mathbb{Q}$ is simple.
   (b) If K_1/F, K_2/F are Galois extensions with Galois group G_1 and G_2, then the composite K_1K_2/F is a Galois extension, with Galois group $G_1 \times G_2$.
   (c) Any finite extension of $\mathbb{F}_p$ is simple.
   (d) Any simple extension of $\mathbb{F}_p$ is finite.
   (e) The Galois group of $\mathbb{Q}(e^{\pi i/8})/\mathbb{Q}$ is the cyclic group of order 15.

Problems for credit

(1) You should have the assignment from another student available for review in your Canvas todo list. Review problem 4 from that assignment (submit your comment on Canvas, *not* with this assignment). Refer to the peer review instructions on the course website for more details. *You are encouraged not to look at the official solution before submitting your review!*
(2) Consider the statement "If F is a field and G_1, G_2 are finite groups that can be realized as Galois groups of extensions of F, then $G_1 \times G_2$ is the Galois group of an extension of F".
   (a) Describe what is wrong with the following "proof" of the statement: Let K_1/F be an extension with Galois group G_1, and K_2/F be an extension with Galois group G_2. Then the composite K_1K_2/F has Galois group $G_1 \times G_2$.
   (b) Show that the statement is false by giving an explicit counterexample.
   (c) Let G be an arbitrary finite group. Show that if there is an extension of $\mathbb{Q}$ with Galois group G, then there is an extension of $\mathbb{Q}$ with Galois group $G \times Z_2$ (where Z_2 is the cyclic group of order 2).
(3) (DF, 14.4.2) Find a primitive generator for $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$ over $\mathbb{Q}$.
(4) (DF, 14.4.9) Suppose K/F is a Galois extension with Galois group G and θ is a primitive element for K: $K = F(\theta)$. For any subgroup H of G, let $f(x) = \prod_{\sigma \in H}(x - \sigma(\theta))$. Show $f(x) \in E[x]$, where E is the fixed field of H in K, and that moreover $f(x)$ is the minimal polynomial for θ over E, and the coefficients of $f(x)$ generate E over F.
(5) (DF, 14.5.5) Let p be a prime and let $\epsilon_1, \ldots, \epsilon_{p-1}$ denote the primitive pth roots of unity in $\mathbb{C}$. Set $p_n = \epsilon_1^n + \ldots + \epsilon_{p-1}^n$. Prove that $p_n = -1$ if p does not divide n, and $p_n = p - 1$ if p does divide n.

Date: April 13, 2020.

(6) (DF, 14.5.8) Let $K_n = \mathbb{Q}(\zeta_{2^{n+2}})$ $(n \geq 0)$. Set $\alpha_n := \zeta_{2^{n+2}} + \zeta_{2^{n+2}}^{-1}$, and let $K_n^+ := \mathbb{Q}(\alpha_n)$.
  (a) Show that $[K_n : \mathbb{Q}] = 2^{n+1}$, $[K_n : K_n^+] = 2$, $[K_n^+ : \mathbb{Q}] = 2^n]$, and $[K_{n+1}^+ : K_n^+] = 2$.
  (b) Determine the minimal polynomial of $\zeta_{2^{n+2}}$ over K_n^+ (in terms of α_n).
  (c) Show that $\alpha_{n+1}^2 = 2 + \alpha_n$, and deduce that:

$$\alpha_n = \sqrt{2 + \sqrt{2 + \sqrt{\ldots + \sqrt{2}}}}$$

  where the square root symbol appears n times. *[This gives an explicit way to realize K_n by iterating quadratic extensions.]*

(7) (DF, 14.5.10) Prove that $\mathbb{Q}(\sqrt[3]{2})$ is not a subfield of any cyclotomic field over $\mathbb{Q}$.