Math-123: finitely generated field extensions

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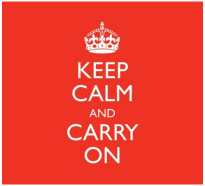
March 11, 2020

Transition to a remote class



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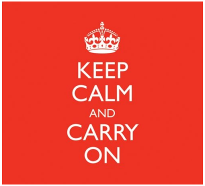

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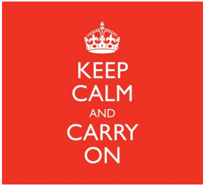

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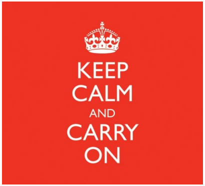

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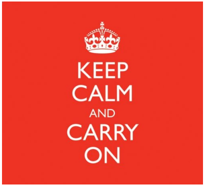

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If K/F is an extension and $\alpha \in K$, $F(\alpha)$ is called a *simple extension* of F. If α is algebraic, the degree of the extension is the degree of the minimal polynomial of α (the monic poly of min degree that α is a root of).

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Example

Example ($K = \mathbb{C}$): $\mathbb{Q}(\sqrt[3]{2})$ has degree 3. The minimal polynomial of $\sqrt[3]{2}$ is $x^3 - 2$.

Finitely generated field extensions

We now want to analyze potentially more complicated extensions, like $\mathbb{Q}(\sqrt{2}, \sqrt[3]{2})$. Such extensions (of the form $F(A)$, for A finite) are called *finitely generated*.

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More precisely, if $K = F(\alpha_1, \ldots, \alpha_n)$, then letting $F_0 = F$, $F_{i+1} = F_i(\alpha_{i+1})$, we get a chain of extensions $F_0 \subseteq F_1 \subseteq \ldots \subseteq F_n$, where $F_i = F(\alpha_1, \ldots, \alpha_i)$. In particular, $F_n = K$.

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Theorem (Multiplicativity of degrees)

If $F \subseteq K \subseteq L$ are field extensions, then $[L : F] = [K : F][L : K]$.

Consequence of multiplicativity of degrees

Thus if we have $K = F(\alpha_1, \ldots, \alpha_n)$, then letting $F_0 = F$, $F_{i+1} = F_i(\alpha_{i+1})$, and the chain of extensions $F_0 \subseteq F_1 \subseteq \ldots \subseteq F_k$ as before, we have:

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Example

It could be strictly less: take $F = \mathbb{Q}$, $\alpha_1 = \sqrt{2}$, $\alpha_2 = \sqrt[6]{2}$. Then $n_1 = 2$, $n_2 = 6$, but $\mathbb{Q}(\alpha_1, \alpha_2) = \mathbb{Q}(\alpha_2)$ has degree $6 < 2 \cdot 6$.

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We saw last time we can also deduce that $[\mathbb{Q}(\sqrt[6]{2}) : \mathbb{Q}(\sqrt{2})] = 3$ (to do this directly, we would have to show $\sqrt[6]{2} \notin \mathbb{Q}(\sqrt{2})$, which is annoying).

Some important conclusions

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[Why? Let $\alpha \in L$. Since α is algebraic over K, α is the root of a polynomial $a_0 + a_1 x + \ldots + a_n x^n$, with each $a_i \in K$. Since K is algebraic over F, each a_i is algebraic over F. Thus the extension $F(a_0, \ldots, a_n)/F$ is finite. The extension $F(a_0, \ldots, a_n)(\alpha)/F(a_0, \ldots, a_n)$ is also finite. Thus $F(a_0, \ldots, a_n, \alpha)$ is finite, so α is algebraic over F.]

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Since $\mathbb{C}$ (or $\mathbb{R}$) are uncountable, this shows there are transcendental elements. However proving specific elements (like e or π) are transcendental is much harder.

Composite fields

How do we "put two fields together"?

Definition

For K_1, K_2 subfields of K, let K_1K_2, the *composite field of* K_1 *and* K_2, be the smallest subfield of K containing K_1 and K_2.

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- $\subseteq$: $\sqrt{2} \in \mathbb{Q}(\sqrt[6]{2})$ as $\sqrt{2} = \sqrt[6]{2}^3$. Similarly, $\sqrt[3]{2} \in \mathbb{Q}(\sqrt[6]{2})$.
- $\supseteq$: $\sqrt[6]{2} = 2^{\frac{1}{6}} = 2^{\frac{1}{2}-\frac{1}{3}} = \frac{\sqrt{2}}{\sqrt[3]{2}} \in \mathbb{Q}(\sqrt{2}, \sqrt[3]{2})$.

Degree of composite extensions

Observe that if K_1, K_2 are finite extensions of F with bases $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_m$, then $K_1K_2 = F(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m)$.

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This implies that $(\alpha_i \beta_j)_{i=1\ldots n, j=1\ldots m}$ spans $K_1 K_2$, so $[K_1 K_2 : F] \leq nm = [K_1 : F][K_2 : F]$.

Degree of composite extensions: picture

If $[K_1 : F] = n$, $[K_2 : F] = m$, then $[K_1K_2 : F] \leq nm$.

$$
\begin{tikzcd}
& K_1K_2 \arrow[dl, dash, "\leq m"'] \arrow[dr, dash, "\leq n"'] & \\
K_1 \arrow[dr, dash, "n"'] & & K_2 \arrow[dl, dash, "m"'] \\
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\end{tikzcd}
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& K_1K_2 & \\
K_1 \arrow[ur, dash, "\leq m"] & & K_2 \arrow[ul, dash, "\leq n"'] \\
& F \arrow[ul, dash, "n"] \arrow[ur, dash, "m"'] &
\end{tikzcd}$$

Note that if $\gcd(n, m) = 1$, then equality holds! $[K_1K_2 : F] = nm$.
We can use this to get another proof that $[\mathbb{Q}(\sqrt{2}, \sqrt[3]{2}) : \mathbb{Q}] = 6$.

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- Thus finite is equivalent to “generated by finitely-many algebraic elements”.
- It follows that algebraic elements form a field, and that the iterations of two algebraic extensions is algebraic.
- We can either think of $F(\alpha, \beta)$ as $(F(\alpha))(\beta)$, or as the composite of the extensions $F(\alpha)$ and $F(\beta)$. If the degrees of these are n and m, we get that the degree of the composite is $\leq nm$, and equality holds if n and m are coprime.