Math-123: straightedge and compass constructions

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[Note: Why is it "straightedge and compass" rather than "ruler and compass"? Because rulers have markings, and we are not allowed to use them. In fact it is possible to trisect an angle using a ruler's markings. See the book!]

Constructible real numbers: geometric definition

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4. If C is an S-constructed circle and L is an S-constructed line which intersects C at a point P, then $P \in S$.

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Example

The point $(2,0)$ is constructible: draw the line L between $(0,0)$ and $(1,0)$, find the intersection with the circle with center $(1,0)$ passing through $(0,0)$. Similarly, $(n,0)$ is constructible for $n \in \mathbb{Z}$.

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It follows that $(0, 1)$ is constructible: draw a line L from $(0, 0)$ to $(1, 0)$, draw the perpendicular line $L^{\perp}$ through $(0, 0)$, and find its intersection with a circle of radius 1 centered at $(0, 0)$.

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- A circle is constructible if and only if its center is a constructible point and its radius is a constructible number.

From geometry to algebra

Lemma

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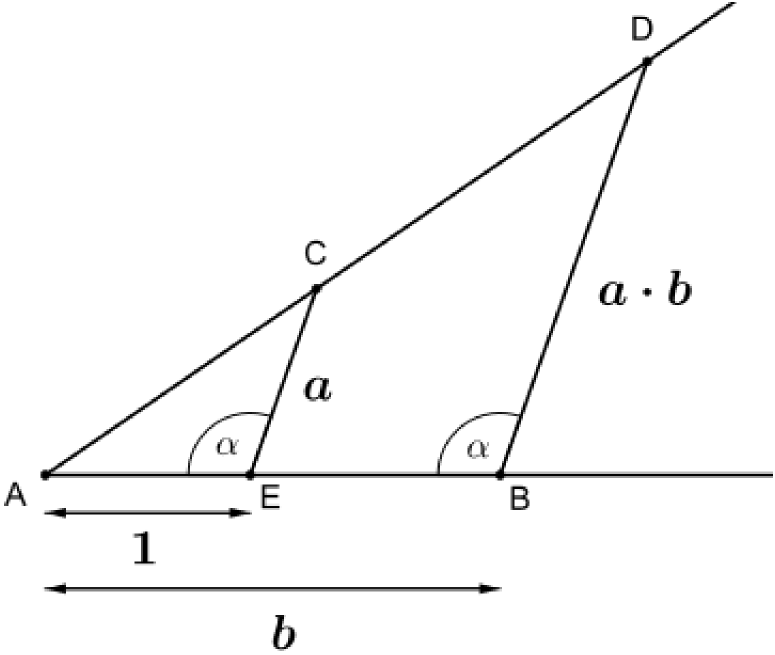
Proof.

$|-a| = a$, and $a + b$ is the distance between $(b, 0)$ and $(-a, 0)$. For products and inverses, use similar triangles. For roots, use a circle of diameter $1 + a$ (pictures to follow). □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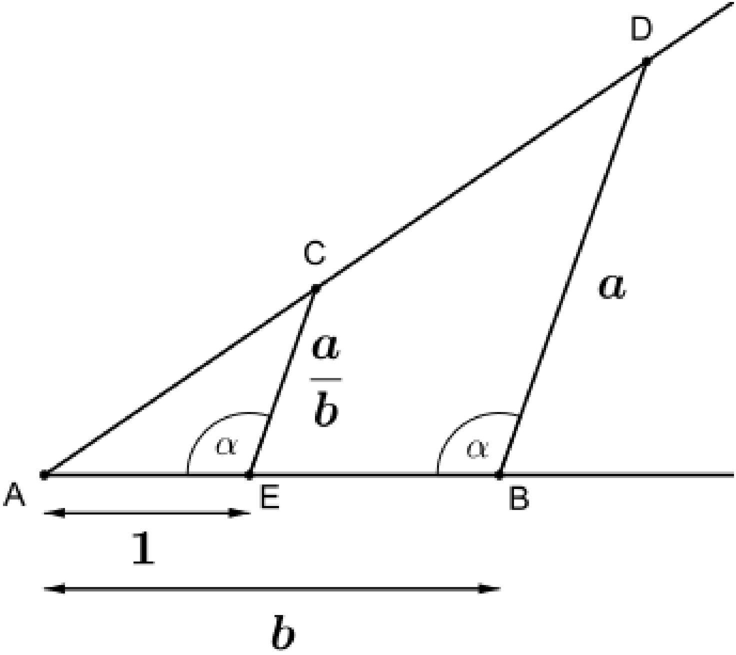
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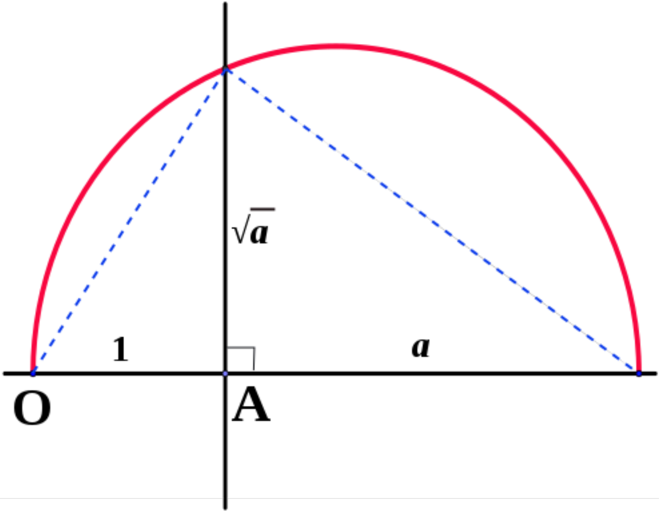

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Theorem

A real number x is constructible if and only if it is contained in an extension K of $\mathbb{Q}$ which is an iteration of quadratic extensions: there are subfields $\mathbb{Q} = F_1 \subseteq F_2 \subseteq \ldots \subseteq F_k = K$ such that $F_1 = \mathbb{Q}$ and for all $i < k$, $F_{i+1} = F_i(\sqrt{\alpha_i})$, for α_i a non-negative real number in F_i.

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Proof sketch.

We just saw that numbers arising from iterated quadratic extensions are constructible. Conversely, any constructible number arises from taking sums, products, and finding root of degree two polynomials using the quadratic formula (to find intersection points of lines and circles). □

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Because the degree of an extension must divide the degree of a bigger extension, we have:

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Thus we cannot double the cube: take a cube of side length 1, which has volume 1. A cube of double that volume would have side length $\sqrt[3]{2}$, which has degree 3, which is not a power of 2.

We also cannot square the circle: a circle of unit radius has area π, and since π (so also $\sqrt{\pi}$) is transcendental, its degree is infinite, hence not a power of 2.

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Thus the problem of trisecting the angle asks whether $\cos(\theta)$ constructible implies $\cos(\theta/3)$ constructible.

Note that $\cos(\theta) = 2\cos^2(\theta/2) - 1$ (double angle formula), so if we know $\cos(\theta)$, we can get $\cos(\theta/2)$ by solving a quadratic. Thus bisecting the angle is possible (try to find a geometric construction!)

Trisecting the angle is impossible

By contrast, $\cos(\theta) = 4\cos^3(\theta/3) - 3\cos(\theta/3)$ (triple angle formula). Set $\theta = \pi/3$ (60 degrees). Recall that $\cos(\theta) = \frac{1}{2}$, so if $\beta = \cos(\theta/3)$, then we must have $4\beta^3 - 3\beta - \frac{1}{2} = 0$, or $8\beta^3 - 6\beta - 1 = 0$.

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Set $\alpha = 2\beta$ to clarify things. Then the above equation becomes $\alpha^3 - 3\alpha - 1 = 0$. The left hand side is an irreducible polynomial in α (it has no rational roots, check it!), so α, hence β, has degree 3, hence is not constructible.

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We just saw that $\cos(\frac{1}{9}\pi)$ is not constructible. Coming up eventually: for which rational numbers r is $\cos(r\pi)$ constructible?

More field theory

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After adding enough roots, we see that every polynomial has a splitting field (13.4.25 in DF). We will also see that splitting fields are unique up to isomorphism, so we talk about "the" splitting field of a polynomial $p(x)$ over F.

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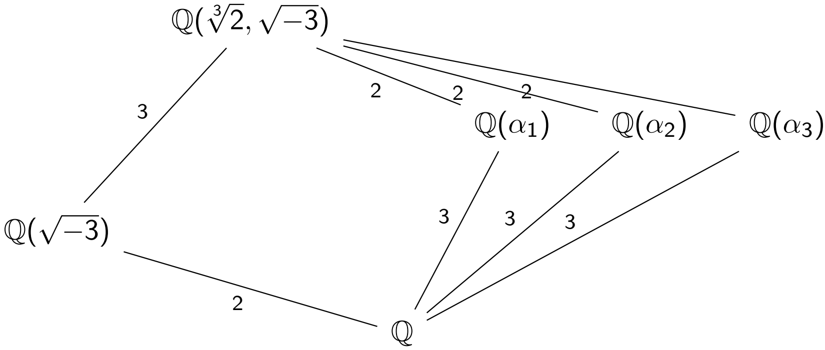
- Instead, observe that $\alpha_2/\alpha_1 = e^{2\pi i/3} \in K$. In cartesian coordinates, it is $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$, so $\sqrt{3}i = \sqrt{-3} \in K$. This is a root of $x^2 + 3$, so has degree 2, hence in fact $K = \mathbb{Q}(\sqrt[3]{2}, \sqrt{-3})$ has degree 6.

Picture of the splitting field of $x^3 - 2$, and known subfields:



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Proof.

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We could have figured out the degree d of the splitting field of $x^3 - 2$ this way: We know $d \leq 3! = 6$, and must strictly contain $\mathbb{Q}(\sqrt[3]{2})$ which has degree 3, so $3 < d \leq 6$, and 3 divides d: $d = 6$ is the only possibility.

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Example

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Consider the polynomial $p(x) = x^4 + 4$. It factors as $(x^2 + 2x + 2)(x^2 - 2x + 2)$. The roots are $\pm 1 \pm i$, so in fact the splitting field is $\mathbb{Q}(i)$, of degree 2!

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- Trisecting an angle θ is equivalent to constructing $\cos(\theta/3)$. This is impossible for $\theta = \pi/3$, because the triple angle formula shows $\cos(\theta/3)$ must be the root of an irreducible polynomial of degree 3.
- The splitting field of a polynomial (over a base) is the smallest field extension containing all the roots of that polynomial. If the polynomial has degree n, the splitting field has degree at most $n!$.