Math-123: Splitting field and algebraic closure

Sebastien Vasey

Harvard University

March 25, 2020

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At least for trisecting the angle we can! See Terrence Tao's proof linked on the course website.

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An extension K of a field F is called a *splitting field* for a polynomial $p(x) \in F[x]$ if $p(x)$ factors into linear factors in $K[x]$ (we say that $p(x)$ *splits completely in* $K[x]$), and does not split completely over any proper subfield of K containing F.

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We say "the" splitting field because it is unique up to isomorphism.

Uniqueness of splitting fields

Lemma (Uniqueness of simple extensions, 13.1.8 in DF)

Let $\phi : F \cong F'$ be an isomorphism. Let $p(x) \in F[x]$ be an irreducible polynomial and let $p'(x) \in F'[x]$ be the polynomial obtained by applying ϕ to the coefficients of p.

Let α be a root of $p(x)$ (in some extension of F) and let α' be a root of $p'(x)$ (in some extension of F'). Then there exists an isomorphism $\sigma : F(\alpha) \cong F'(\alpha')$ such that $\sigma \restriction F = \phi$.

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Theorem (Uniqueness of splitting field, 13.1.27 in DF)

Let $\phi : F \cong F'$ be an isomorphism. Let $p(x) \in F[x]$ be a polynomial and let $p'(x) \in F'[x]$ be the corresponding polynomial. Let K be a splitting field for $p(x)$ over F, and let K' be a splitting field for $p'(x)$ over F'. Then ϕ extends to $\sigma : K \cong K'$.

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Proof.

By induction on the degree, n, of $p(x)$. If $n = 1$, $F = K$ and $F' = K'$, so we can take $\sigma = \phi$.

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Proof.

By induction on the degree, n, of $p(x)$. If $n = 1$, $F = K$ and $F' = K'$, so we can take $\sigma = \phi$. If $n \geq 2$, let $f(x)$ be an irreducible factor of $p(x)$. Add a root $\alpha \in K$ for f, $\alpha' \in K'$ for f'. Get $\phi' : F(\alpha) \cong F(\alpha')$. Apply the IH to $p(x)/(x - \alpha)$. □

That is, apply the induction hypothesis to the top part of this diagram and the polynomial $p(x)/(x-\alpha)$, of degree $n-1$.

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Note the special case where $F = F'$ and ϕ is the identity. Then we get that any two splitting fields of $p(x)$ over F are isomorphic (and the isomorphism fixes the elements of F).

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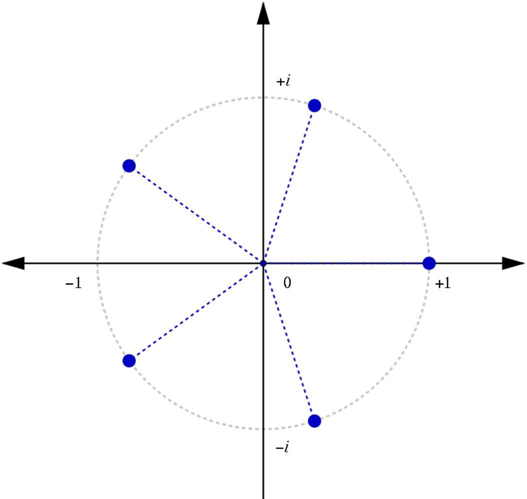

Roots of unity

We usually consider the roots of unity inside $\mathbb{C}$, but we can more generally look at them inside *any* field F.

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$F^\times$ is abelian, so G is also abelian. By the structure theorem for finitely generated $\mathbb{Z}$-modules, G is isomorphic to $Z_{n_1} \times Z_{n_2} \times Z_{n_3} \times \ldots \times Z_{n_k}$, where $2 \leq n_1 | n_2 | \ldots | n_k$, and we have written $Z_\ell = \mathbb{Z}/\ell\mathbb{Z}$.

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In particular, any member of G is a root of the polynomial $x^{n_k} - 1$. Thus $n_1 \cdot n_2 \cdot \ldots \cdot n_k = |G| \leq n_k$, so $k = 1$, so G is cyclic. □

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Definition

The field $\mathbb{Q}(\zeta_n)$ is called the *cyclotomic field of nth roots of unity*.

Degree of the cyclotomic field

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We can prove it now for $n = p$ a prime. First factor:

$$x^p - 1 = (x-1)(x^{p-1} + x^{p-2} + \ldots + 1)$$

Since $\zeta_p \neq 1$, it is a root of $f(x) = x^{p-1} + x^{p-2} + \ldots + 1$. This polynomial is irreducible, so ζ_p has degree $p - 1 = \phi(p)$.

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Why is $f(x)$ irreducible? Observe $f(x) = \frac{x^p - 1}{x - 1}$. Replace x by $x + 1$, and write $\binom{p}{k} := \frac{p!}{k!(p-k)!}$. By the binomial theorem, get:

$$\frac{1}{x}\left(\binom{p}{0}x^p + \binom{p}{1}x^{p-1} + \binom{p}{2}x^{p-2} + \ldots + \binom{p}{p-1}x\right)$$
$$= x^{p-1} + \binom{p}{1}x^{p-2} + \binom{p}{2}x^{p-3} + \ldots + \binom{p}{p-1}$$

Note p divides all the non-leading coefficients, but the last coefficient is p. Apply Eisenstein's criterion.

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What is the degree of the splitting field? Well $\mathbb{Q}(\sqrt[p]{2}, \zeta_p)$ is the composite of $\mathbb{Q}(\sqrt[p]{2})$ and $\mathbb{Q}(\zeta_p)$. The first has degree p, the second degree $p - 1$. Since p and $p - 1$ are relatively prime, the degree of the splitting field is $p(p - 1)$.

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Note that if K is algebraically closed, then every $p(x) \in K[x]$ has *all* its roots in K: use repeated division.

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Another way to think about it: we fix once and for all an extension K of F containing an algebraic closure $\bar{F}$ of F, then can assume all roots of polynomial in $F[x]$ are in K. We did this already for $F = \mathbb{Q}$ by working in $K = \mathbb{C}$.

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The number n_i is called the *multiplicity* of the root α_i. If $n_i = 1$, α_i is called a *simple root*. Otherwise it is called a *multiple root*.

Multiplicity of roots

Let F be a field, $f(x) \in F[x]$ be a polynomial with leading coefficient $a_n \neq 0$.

In the splitting field K of $f(x)$ over F, we can write:

$$f(x) = a_n(x - \alpha_1)^{n_1}(x - \alpha_2)^{n_2} \dots (x - \alpha_k)^{n_k}$$

where $\alpha_1, \dots, \alpha_k \in K$ are distinct, and $n_i \geq 1$ for all i.

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Definition

We call $f(x)$ *separable* if it has no multiple roots (i.e. $n_i = 1$ for all i). We call $f(x)$ *inseparable* otherwise.

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- A nontrivial example: take $F = \mathbb{F}_2(t)$, the field of rational functions in t. Consider $x^2 - t \in F[x]$. It is irreducible (!) by Eisenstein: t is a prime element of $\mathbb{F}_2[t]$.
  Let $\sqrt{t}$ denote a root (in some extension). Then
  $(x - \sqrt{t})^2 = x^2 + t = x^2 - t$ (because $2 = 0$ in this field!).
  Thus $x^2 - t$ is inseparable: $\sqrt{t}$ has multiplicity 2.

Testing for multiple roots

Definition

If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0 \in F[x]$, the *derivative* of $f(x)$ is the polynomial
$D_x f(x) := n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \ldots + 2a_2 x + a_1 \in F[x]$.

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Exercise: check that the sum and product rules for derivatives hold in this context.

Theorem

A polynomial $f(x)$ has multiple root α if and only if α is a root of both $f(x)$ and $D_x f(x)$.

Corollary

$f(x)$ has multiple root α if and only if $f(x)$ and $D_x f(x)$ are both divisible by the minimal polynomial of α (over the base field). In particular, $f(x)$ is separable if and only if it is coprime to $D_x f(x)$.

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Proof.

If f has degree $n \geq 1$, then $D_x f(x)$ has degree $n - 1$. In particular, it is not zero. The only divisors of f are 1 and $f(x)$, and by degree consideration, $f(x)$ does not divide $D_x f(x)$. □

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Question to think about: where does this fail in characteristic p?

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Question to think about: where does this fail in characteristic p? We will talk more about it next time.

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Assume α is a root of $f(x)$ of multiplicity n. Then $f(x) = (x - \alpha)^n g(x)$ (in a splitting field), for some $n \geq 1$. Take derivatives, get $D_x f(x) = n(x - \alpha)^{n-1} g(x) + (x - \alpha)^n D_x g(x)$.

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If $n = 1$, then $D_x f(x) = g(x) + (x-\alpha)^n D_x g(x)$. Evaluating at α, we get $g(\alpha)$. By definition of multiplicity, $g(\alpha) \neq 0$, so α is not a root of $D_x f(x)$. □

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- A field is *algebraically closed* if all polynomials factor into linear terms. The complex numbers are algebraically closed.
- The *algebraic closure* is the smallest algebraically closed extension of a given field. Every field has a unique algebraic closure.
- A polynomial is *separable* if it has no multiple roots, equivalently if it is coprime to its derivative. In characteristic zero, irreducible implies separable.