Math-123: Basic definitions of Galois theory

Sebastien Vasey

Harvard University

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If you are below 70%, you may or may not get SEM depending on class performance, participation, special circumstances, etc.

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We proved that $\Phi_n(x)$ is a monic polynomial of degree $\phi(n)$, with integer coefficients.

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Say $g(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$. By properties of the Frobenius map, $g(x)^p = a_n^p x^{pn} + a_{n-1}^p x^{p(n-1)} + \ldots + a_0^p$.

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Now use that $a^p = a$ for every $a \in \mathbb{F}_p$ (Lagrange's theorem applied to the group of units $\mathbb{F}_p^\times$ – also called Fermat's little theorem).

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So $\bar{f}$ and $\bar{g}$ have an irreducible factor in common in $\mathbb{F}_p[x]$!

Proof of irreducibility of Φ_n, continued

We know also $\bar{f}\bar{g} = \overline{\Phi_n}(x)$. Since $\bar{f}$ and $\bar{g}$ have a factor in common, $\overline{\Phi_n}(x)$ has a multiple root in $\mathbb{F}_p[x]$.

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Continuing in this way, we get that $f(\zeta^{p_1 p_2 \cdots p_m}) = 0$. Thus f has *all* primitive roots of unity as roots, so $f = \Phi_n$, as desired.

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- An automorphism σ of K *fixes* an element a if $\sigma(a) = a$. We say σ *fixes* a set A if $\sigma(a) = a$ for all $a \in A$.
- If K is an extension of F, we write $\mathrm{Aut}(K/F)$ for the set of all automorphisms of K which fix F.

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The second observation says that automorphisms *permute* the roots of a polynomial. Abstractly: the group $\text{Aut}(K/F)$ *acts* on these roots.

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  - If $\tau(\sqrt{2}) = \sqrt{2}$, τ is the identity. On the other hand $\tau(\sqrt{2}) = -\sqrt{2}$ is an automorphism (can check directly, or use uniqueness of simple extensions).
  - Thus $\mathrm{Aut}(K/F) = \{1, \sigma\}$, where σ sends $\sqrt{2}$ to $-\sqrt{2}$. It is the cyclic group of order 2.

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This shows that $\mathrm{Aut}(K/F) = \{1\}$, the trivial group.

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(one can check it is indeed a subfield: if a and b are fixed, ab, $a+b$, a/b are fixed too!).

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2. If $H_1 \subseteq H_2 \subseteq \text{Aut}(K)$, with fixed fields F_1, F_2, then $F_2 \subseteq F_1$ [Fewer automorphisms fix more things].

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So in this case, $\mathbb{Q}$ is not the fixed field of any subgroup. Intuitively, we are "missing roots" for $x^3 - 2$.

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Proof: We more generally ask: given an isomorphism $\phi : F \cong F'$, K a splitting field of $f(x)$, K' a splitting field of the corresponding polynomial $f' = \phi(f)$, how many isomorphisms $\sigma : K \cong K'$ does ϕ extend to?

$$\begin{array}{ccc} K & \xrightarrow[\sigma]{\cong} & K' \\ | & & | \\ F & \xrightarrow[\phi]{\cong} & F' \end{array}$$

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Let K be the splitting field of a polynomial $f(x) \in F[x]$. Then $\mathrm{Aut}(K/F)$ has at most $[K : F]$ elements, with equality if f is separable.

Proof: We more generally ask: given an isomorphism $\phi : F \cong F'$, K a splitting field of $f(x)$, K' a splitting field of the corresponding polynomial $f' = \phi(f)$, how many isomorphisms $\sigma : K \cong K'$ does ϕ extend to?

$$\begin{array}{ccc} K & \xrightarrow[\sigma]{\cong} & K' \\ | & & | \\ F & \xrightarrow[\phi]{\cong} & F' \end{array}$$

The special case we care about is when $F = F'$, ϕ is the identity, $K = K'$.

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By the induction hypothesis, there are (at most) $[K : F(\alpha)]$-many ways to extend each ϕ' to a σ, with equality if $\frac{f(x)}{x-\alpha}$ is separable.

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In total, there are at most $[F(\alpha) : F] \cdot [K : F(\alpha)] = [K : F]$ extensions, with equality if $f(x)$ is separable.

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A finite extension K/F is *Galois* if $|\mathrm{Aut}(K/F)| = [K : F]$. If K/F is Galois, we call $\mathrm{Aut}(K/F)$ the *Galois group* of K/F.

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In general, the splitting field of *any* polynomial in $\mathbb{Q}[x]$ is Galois over $\mathbb{Q}$: consider the product of its distinct irreducible factors.

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5. The Galois group is $\{1, \sigma, \tau, \sigma\tau\}$. All elements have order 2, so it is isomorphic to $Z_2 \times Z_2$ (the Klein 4-group).

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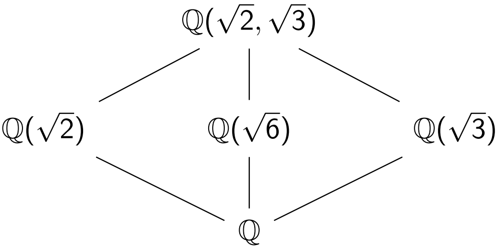
$\{1, \sigma\tau\}$	$\mathbb{Q}(\sqrt{6})$
$\{1, \tau\}$	$\mathbb{Q}(\sqrt{2})$
$\{1, \sigma, \tau, \sigma\tau\}$	$\mathbb{Q}$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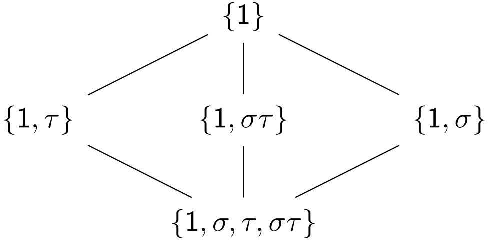
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Any automorphism of K/F must permute the roots of $x^3 - 2$. There are $3! = 6$ such permutations and we know $|\mathrm{Aut}(K/F)| = [K : F] = 6$. Thus any permutation of the roots induces an automorphism, and $\mathrm{Aut}(K/F) \cong S_3$.

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Another way to think about it: It suffices to describe the automorphism on the generators $\sqrt[3]{2}$ and ρ.

$\sqrt[3]{2}$ can be sent to a root of $x^3 - 2$, ρ can be sent to ρ or ρ^2. All these possibilities give automorphisms by counting.

Let $\sigma : \sqrt[3]{2} \mapsto \rho\sqrt[3]{2}$, $\rho \mapsto \rho$, and $\tau : \sqrt[3]{2} \mapsto \sqrt[3]{2}$, $\rho \mapsto \rho^2$.

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We can also determine the fixed fields...

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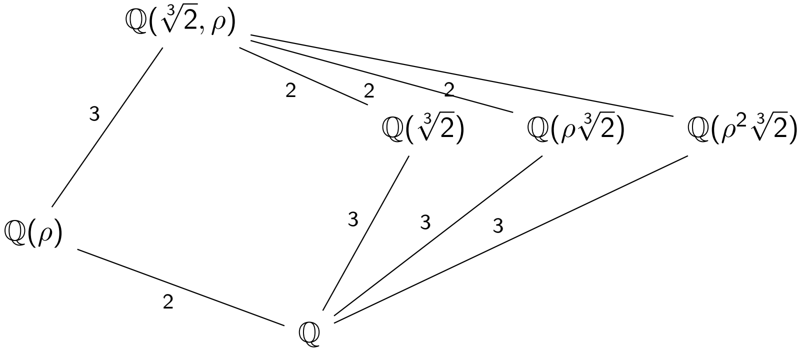

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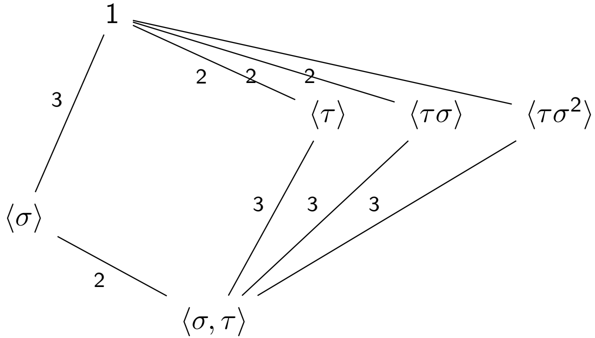

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Thus σ has order n, so the Galois group of F/K is cyclic of order n, generated by the Frobenius map.

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- *[That last one is an April's fool]*.