Math-123: The fundamental theorem of Galois theory, part II

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Subgroups of $\mathrm{Aut}(K/F)$ have a corresponding *fixed field*.

The fundamental theorem, part I (last time)

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2. $[K : E] = |H|$ and $[E : F] = |G : H|$.
3. K/E is always Galois, with Galois group $\text{Aut}(K/E) = H$.

The fundamental theorem: picture

$$K = \text{ fixed field of } 1$$

$$\Big|\, |H| = |H{:}1|$$

$$E = \text{ fixed field of } H$$

$$\Big|\, |G{:}H|$$

$$F = \text{ fixed field of } G$$

$$1 = \text{ automorphisms fixing } K$$

$$\Big|\, [K{:}E]$$

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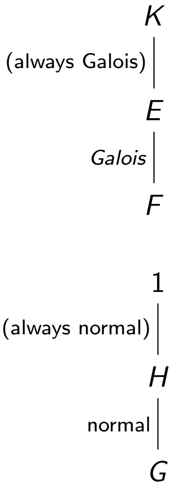
Let K/F be a Galois extension with Galois group $G = \mathrm{Aut}(K/F)$.
Let E be an intermediate field ($F \subseteq E \subseteq K$), with corresponding group $H = \mathrm{Aut}(K/E)$.
Then E is Galois over F if and only if H is a normal subgroup of G. In this case, $\mathrm{Aut}(E/F) \cong G/H$.

The fundamental theorem, part II: picture



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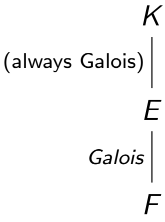
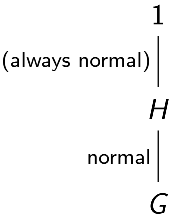
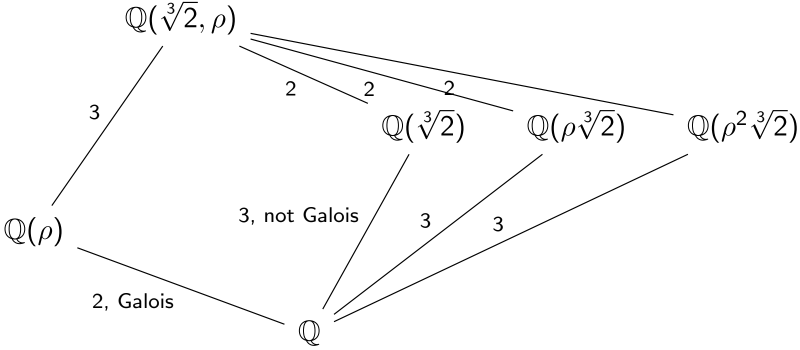




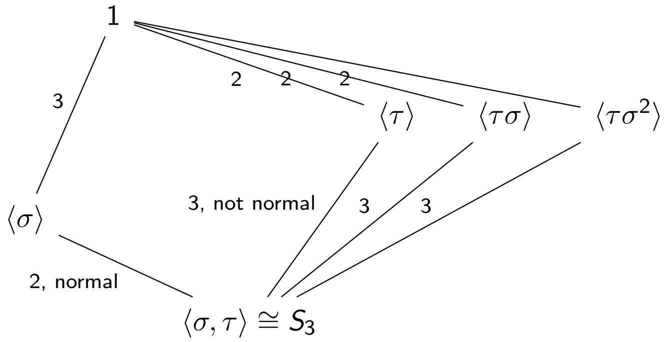
If H is normal in G, then
$G/H = \mathrm{Aut}(K/F)/\mathrm{Aut}(K/E) \cong \mathrm{Aut}(E/F)$.

Part II in action, splitting fields of $x^3 - 2$

$(\rho = e^{2\pi i/3})$



Part II in action, splitting field of $x^3 - 2$: group side



Proof of part II: Fix a subgroup H of G, let E be its fixed field. How do the members of $\mathrm{Aut}(E/F)$ relate to members of $\mathrm{Aut}(K/F)$?

Given $\sigma \in \mathrm{Aut}(K/F)$, $\sigma \restriction E$ may not be in $\mathrm{Aut}(E/F)$ (maybe $\sigma[E] \neq E$). What is true is that $\sigma \restriction E$ is an embedding ($=$ injective homomorphism) of E into K, fixing F

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As K/F is Galois, K is the splitting field of a separable polynomial $f(x) \in F[x]$. K is the splitting field of $f(x) \in E[x]$ and the splitting field of $\tau(f(x)) = f(x) \in \tau[E][x]$. By results about splitting fields, τ extends to $\sigma \in \mathrm{Aut}(K/F)$.

$$\begin{array}{ccc} K & \xrightarrow[\sigma]{\cong} & K \\ \Big| & & \Big| \\ E & \xrightarrow[\tau]{\cong} & \tau[E] \end{array}$$

So let $\mathrm{Emb}(E/F)$ denote the set of embeddings of E into K fixing F. We showed $\sigma \mapsto \sigma \restriction E$ gives a surjection from $G = \mathrm{Aut}(K/F)$ onto $\mathrm{Emb}(E/F)$.

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We saw the members of $\mathrm{Aut}(E/F)$ are in bijections with cosets of H in G and this bijection respects composition, so gives an isomorphism of $\mathrm{Aut}(E/F)$ with G/H. This concludes the proof of part II!

Fundamental theorem, part III

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If E_1, E_2 correspond to H_1, H_2, then $E_1 \cap E_2$ corresponds to $\langle H_1, H_2 \rangle$, and $E_1 E_2$ corresponds to $H_1 \cap H_2$.

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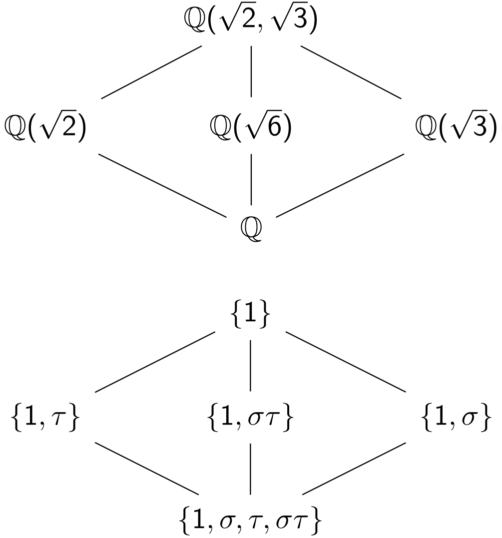
Proof.

Exercise! Use the definition of the fixed field. $\square$

Part III in action: $\mathbb{Q}(\sqrt{2}, \sqrt{3})$



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In particular, this polynomial is irreducible, and $\mathbb{Q}(\sqrt{2}+\sqrt{3})$ has degree 4, so is equal to $\mathbb{Q}(\sqrt{2},\sqrt{3})$.

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This has degree at most $2 \cdot 8 = 16$, but strictly more than 8 (i is not real), so must have degree 16.

Splitting field of $x^8 - 2$, continued

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This determines the group completely: it is a "quasidihedral group" of order 16. See the book for the computation of the subgroups and subfields.

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More precisely: if σ is the Frobenius map, d a divisor of n, H the subgroup generated by σ^d, then $|H| = \frac{n}{d}$, so if E is the fixed field, $[K : E] = \frac{n}{d}$ and $[E : F] = d$.

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Since cyclic groups are abelian, all the subgroups are normal, so E/F is Galois (which we knew already).

Irreducible polynomials over $\mathbb{F}_p$

Remember that in the assignments you built some finite fields by hand by exhibiting irreducible elements in $\mathbb{F}_p[x]$ of certain degrees.

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The extension $\mathbb{F}_{p^n}/\mathbb{F}_p$ is simple: $\mathbb{F}_{p^n} = \mathbb{F}_p(\theta)$ for some θ. In particular, the minimal polynomial of θ is irreducible in $\mathbb{F}_p[x]$ of degree n.

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Proof.

We say any finite subgroup of the group of units of a field is cyclic, so $\mathbb{F}_{p^n}^{\times}$ is cyclic: take θ to be a generator. □

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Conversely, if $p(x) \in \mathbb{F}_p[x]$ is any irreducible polynomial of degree d which divides $x^{p^n} - x$, and $p(\alpha) = 0$, then $\mathbb{F}_p(\alpha)$ is a subfield of $\mathbb{F}_{p^n}$ of degree d.

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Putting all of this together: $x^{p^n} - x$ is the product of $(x - \beta)$, for β a root. β has a certain minimal polynomial of degree d. That degree must divide n. Conversely, any irreducible poly with degree d dividing n must generate $\mathbb{F}_{p^d} \subseteq \mathbb{F}_{p^n}$, so divides $x^{p^n} - x$.

Theorem

$x^{p^n} - x$ is the product of all the distinct irreducible polynomials in $\mathbb{F}_p[x]$ of degree d, where d runs through all the divisors of n.

This can be used to produce irreducible polynomials recursively, count how many there are, etc. (see DF)

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If $x^4 + 1$ is irreducible over $\mathbb{F}_p[x]$, that would mean it generates an extension K of degree 4, with $\mathbb{F}_p \subseteq K \subseteq \mathbb{F}_{p^2}$. However $\mathbb{F}_{p^2}/\mathbb{F}_p$ has degree 2, contradiction.

Summary

- Fundamental theorem, part II: If K/F is Galois and E is an intermediate field, E/F is Galois if and only if $\mathrm{Aut}(K/E)$ is normal in $\mathrm{Aut}(K/F)$. In this case, $\mathrm{Aut}(E/F) \cong \mathrm{Aut}(K/F)/\mathrm{Aut}(K/E)$.

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- The Galois group of $\mathbb{F}_{p^n}/\mathbb{F}_p$ is cyclic of order n, generated by the Frobenius map. Thus $\mathbb{F}_{p^d}$ is the only subfield of $\mathbb{F}_{p^n}$, for d a divisor of n.