Math-123: Finite fields, Composite extensions

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More precisely: if σ is the Frobenius map, d a divisor of n, H the subgroup generated by σ^d, then $|H| = \frac{n}{d}$, so if E is the fixed field, $[K : E] = \frac{n}{d}$ and $[E : F] = d$.

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Since cyclic groups are abelian, all the subgroups are normal, so E/F is Galois (which we knew already).

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The extension $\mathbb{F}_{p^n}/\mathbb{F}_p$ is simple: $\mathbb{F}_{p^n} = \mathbb{F}_p(\theta)$ for some θ. In particular, the minimal polynomial of θ is irreducible in $\mathbb{F}_p[x]$ of degree n.

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Proof.

We say any finite subgroup of the group of units of a field is cyclic, so $\mathbb{F}_{p^n}^{\times}$ is cyclic: take θ to be a generator. □

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Putting all of this together: $x^{p^n} - x$ is the product of $(x - \beta)$, for β a root. β has a certain minimal polynomial of degree d. That degree must divide n. Conversely, any irreducible poly with degree d dividing n must generate $\mathbb{F}_{p^d} \subseteq \mathbb{F}_{p^n}$, so divides $x^{p^n} - x$.

Theorem

$x^{p^n} - x$ is the product of all the distinct irreducible polynomials in $\mathbb{F}_p[x]$ of degree d, where d runs through all the divisors of n.

This can be used to produce irreducible polynomials recursively, count how many there are, etc. (see DF)

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We have: $x^4 + 1 | x^8 - 1 | x^{p^2-1} - 1 | x^{p^2} - x$. Thus all roots of $x^4 + 1$ are roots of $x^{p^2} - x$, so are in $\mathbb{F}_{p^2}$.

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If $x^4 + 1$ were irreducible over $\mathbb{F}_p[x]$, then it would generate an extension K of degree 4, with $\mathbb{F}_p \subseteq K \subseteq \mathbb{F}_{p^2}$. However $\mathbb{F}_{p^2}/\mathbb{F}_p$ has degree 2, contradiction.

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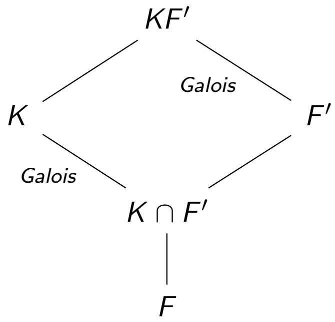
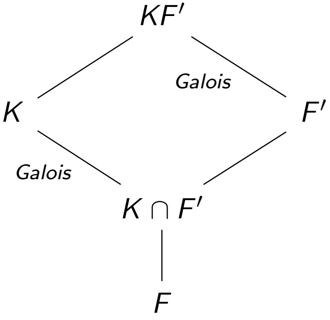
The Galois group is $\mathrm{Aut}(KF'/F') \cong \mathrm{Aut}(K/K \cap F')$.

Galois group of composite extensions: picture



$\mathrm{Aut}(K/(K \cap F')) \cong \mathrm{Aut}(KF'/F')$



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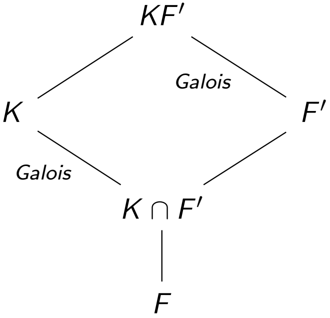

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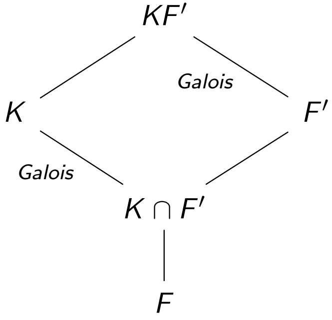

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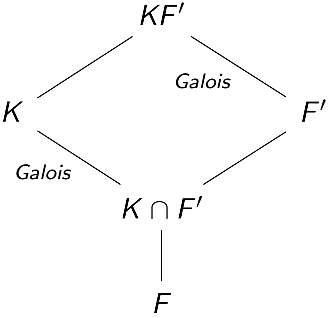

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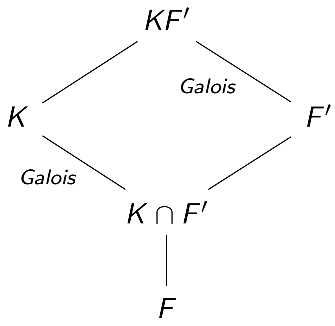
The elements of the kernel fix both K and F', hence fix KF'. Thus the kernel is trivial: ϕ is injective.



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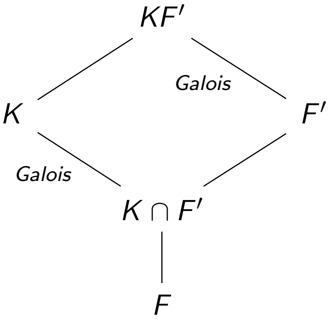

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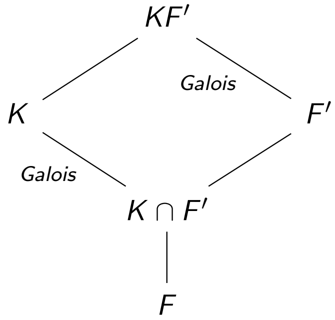
Let H be the image of ϕ. Let K_H be the fixed field of H (in K/F). Every element of H fixes F', so $F' \cap K \subseteq K_H$.



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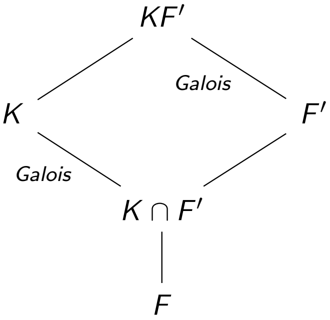

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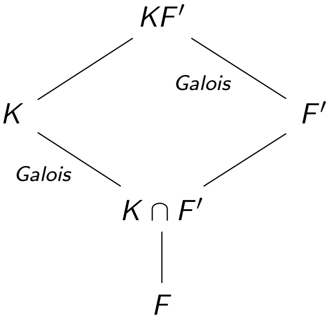

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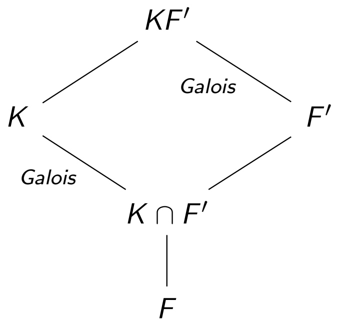

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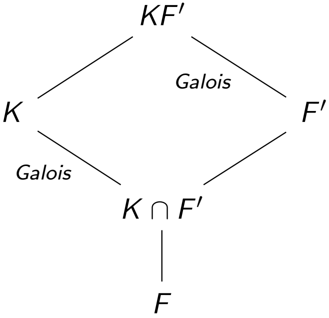
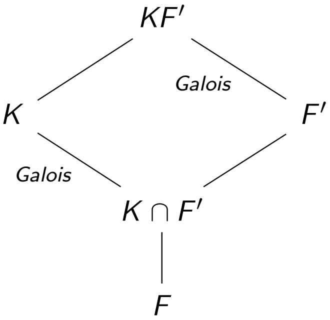
Thus $H = \mathrm{Aut}(K/K \cap F')$, and we are done (first iso theorem).



Corollary

$$[KF' : F] = \frac{[K : F][F' : F]}{[K \cap F' : F]}$$



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Proof.

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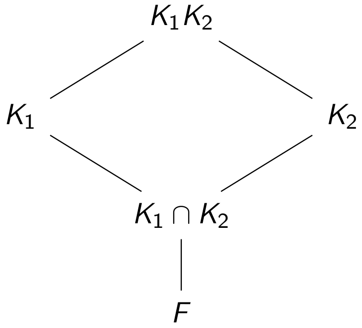
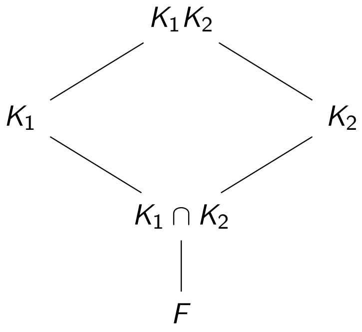
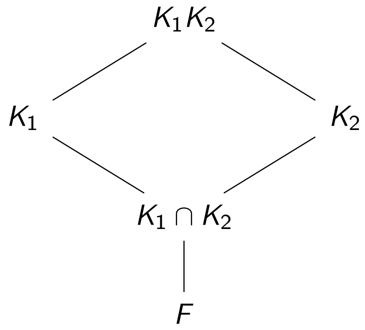




Proof that $(K_1 \cap K_2)/F$ **is Galois**: Let $p(x) \in F[x]$ be irreducible with a root α in $K_1 \cap K_2$.



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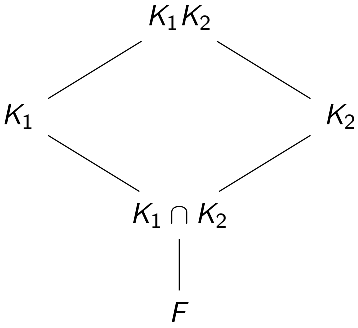
All the roots of $p(x)$ lie in K_1 *and* in K_2 (characterization of Galois extensions). Thus all the roots of $p(x)$ lie in $K_1 \cap K_2$. Thus $K_1 \cap K_2/F$ is Galois.



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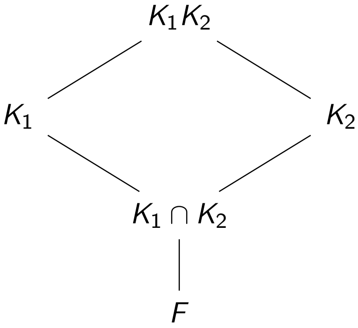

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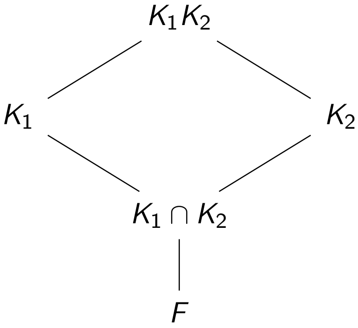
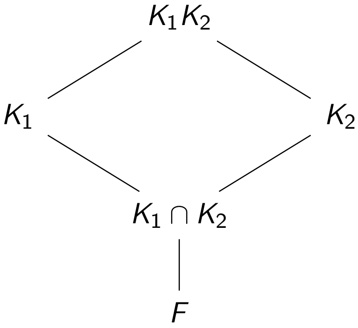
Then K_1K_2 is the splitting field of $f_1(x)f_2(x)$. Removing repeated irreducible factors, we get that K_1K_2 is the splitting field of a separable polynomial.



Description of the Galois group of K_1K_2/F: Consider $\phi : \mathrm{Aut}(K_1K_2/F) \to \mathrm{Aut}(K_1/F) \times \mathrm{Aut}(K_2/F)$ given by $\phi(\sigma) = (\sigma \restriction K_1, \sigma \restriction K_2)$.



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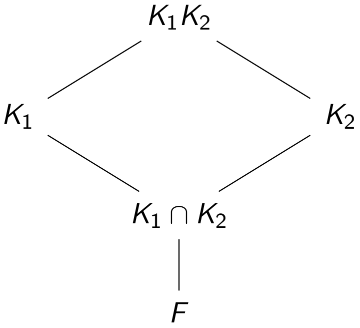

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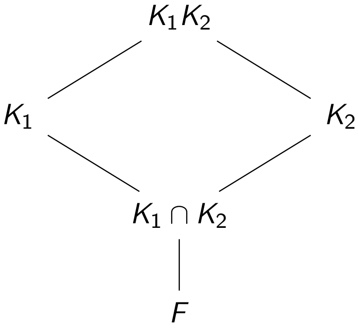

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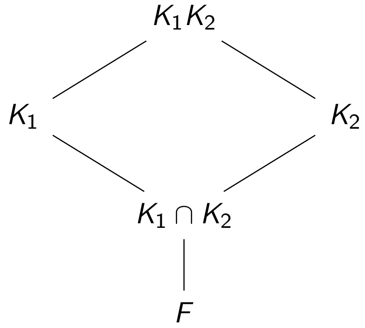

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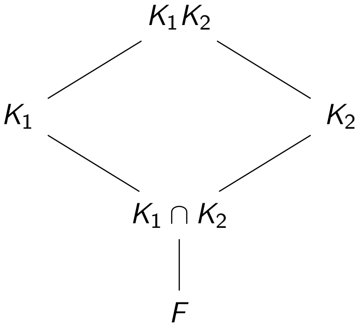

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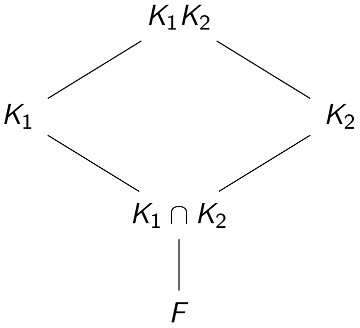

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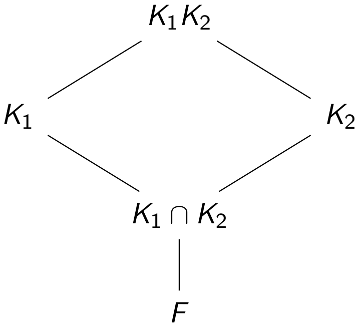

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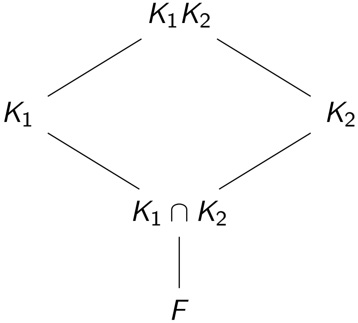

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Using the previous corollary, $|H| = [K_1K_2 : F]$, as desired.

Corollary

Let K_1/F, K_2/F be Galois extensions. If $K_1 \cap K_2 = F$, then $\mathrm{Aut}(K_1 K_2/F) \cong \mathrm{Aut}(K_1/F) \times \mathrm{Aut}(K_2/F)$.

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So $\mathrm{Aut}(K_1K_2/F) = \mathrm{Aut}(\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}) \cong Z_2 \times Z_2$.

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Thus *any algebraic extension of a perfect field is separable.*

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If E/F is any finite separable extension, then E is contained in an extension K which is Galois over F, and is minimal (no proper subfield of K containing E is Galois over F).

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The Galois extension K is called the *Galois closure of E over F*.

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