Math-123: Insolvability of the quintic

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Today we study the following:

Question

If $f(x) \in \mathbb{Q}[x]$, when is there a formula for the roots of f(x) using just addition, multiplication, and extraction of roots?

We first study adjunctions of *n*th roots.

Definition

An extension K/F is a simple radical extension if it is obtained (for some *n*) by adjoining the *n*th root of an element *a* of *F*: K = F(b), where $b^n = a$.

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Note: such an extension K/F will be Galois if and only if it contains *all* the roots of $x^n - a$ if and only if F contains all the *n*th roots of unity.

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This explains why $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is Galois, but $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is not: -1, 1 are the square roots of unity, but \mathbb{Q} does not contain all cube roots of unity.

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Question: What about the converse?

Let K/F be a cyclic extension of degree *n*. If the characteristic of *F* does not divide *n* and *F* contains the *n*th roots of unity, then $K = F(\sqrt[n]{a})$ for some $a \in F$.

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Definition

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$$(\alpha,\zeta) := \alpha + \zeta \sigma(\alpha) + \zeta^2 \sigma^2(\alpha) + \ldots + \zeta^{n-1} \sigma^{n-1}(\alpha)$$

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By the fundamental theorem of Galois theory, $F((\alpha, \zeta))$ is not a proper subfield of K: $F((\alpha, \zeta)) = K$. This completes the proof.

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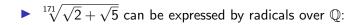
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Definition

A polynomial $f(x) \in F[x]$ can be solved by radicals if all its roots can be expressed by radicals.





Example

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 can be expressed by radicals over \mathbb{Q} : take $K_0 = \mathbb{Q}, K_1 = \mathbb{Q}(\sqrt{2}), K_2 = K_1(\sqrt{5}), K_3 = K_2(\sqrt[171]{\sqrt{2} + \sqrt{5}}).$

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 ³√2 can be expressed by radicals over Q, but is not constructible.

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Let $F = K_0 \subseteq K_1 \ldots \subseteq K_s = K$ be a root extension, and let K'/F be a simple radical extension: $K' = F(\sqrt[n]{a})$.

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Say $F = K_0 \subseteq ... \subseteq K_s = K$ is a root extension, K'/F is another root extension. Then $F \subseteq K_0K' \subseteq ... \subseteq K_sK'$ is an iteration of root extensions, hence a root extension.

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If $\sigma \in \operatorname{Aut}(L/F)$, then we get a chain of subfields $F = K_0 = \sigma[K_0] \subseteq \sigma[K_1] \subseteq \ldots \subseteq \sigma[K_s] = \sigma[K]$.

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For each *i*, $F'K_{i+1}/F'K_i$ is a simple radical extension where the base contains the relevant roots of unity so it is a cyclic extension. F'/F is a composite of cyclotomic extensions, hence abelian, so can be written as an iteration of cyclic extensions. Done!

In conclusion, we have shown:

Theorem

If K/F is a root extension, then there is an extension L of K such that:

1. L/F is Galois.

2. There exists subfield $F = L_0 \subseteq L_1 \subseteq \ldots \subseteq L_s = L$ such that L_{i+1}/L_i is a cyclic extension.

Definition

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Exercise 2: Show that "cyclic" can be replaced by "abelian" in the definition.

Exercise 3: The alternating group A_n and the symmetric group S_n are solvable if and only if $n \le 4$ (use that A_n is simple for $n \ge 5$ — see DF).

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If a polynomial in $\mathbb{Q}[x]$ has Galois group S_n for $n \ge 5$, then it cannot be solved by radicals.

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We will also see that the polynomial $f(x) = x^5 - 6x + 3$ has Galois group S_5 , so is a specific example that cannot be solved by radicals.

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Corollary

If a polynomial in $\mathbb{Q}[x]$ has Galois group S_n for $n \ge 5$, then it cannot be solved by radicals.

We will also see that the polynomial $f(x) = x^5 - 6x + 3$ has Galois group S_5 , so is a specific example that cannot be solved by radicals. On the other hand, any polynomial of degree 4 or less can be solved by radicals.

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The polynomial $f(x) \in F[x]$ can be solved by radicals if and only if the Galois group of f(x) is solvable.

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The splitting field of f(x) is a subfield of L, hence its Galois group is a quotient of G_0 , hence solvable.

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Proof of \Leftarrow : Assume the Galois group G of f(x) is solvable. Let K/F be the splitting field of f(x). Let $1 = G_s \subseteq G_{s-1} \ldots \subseteq G_0 = G$ witness solvability, and let K_i be the fixed field of G_i .

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By previous work, $\operatorname{Aut}(F'K_{i+1}/F'K_i) \cong \operatorname{Aut}(K_{i+1}/K_{i+1} \cap F')$ is a subgroup of $\operatorname{Aut}(K_{i+1}/K_i)$, hence cyclic. So $F'K_{i+1}/F'K_i$ is a cyclic extension.

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As F' contains the relevant roots of unity, $F'K_{i+1}/F'K_i$ is a simple radical extension. As before F'/F is a root extension. We're done.

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- ► The derivative is f'(x) = 5x⁴ 6. It has only two real roots. If f(x) had more real roots, f'(x) would have more real roots (draw a picture).
- Thus f(x) has three real roots, two complex roots. The two complex roots must be conjugate (not preserved by complex conjugation).

So far we know: f(x) is irreducible with three real roots and two complex roots, interchanged by complex conjugation.

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- Therefore the Galois group of f(x) is S_5 .

(In characteristic zero)

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- ► In particular, any polynomial of degree 4 or less can be solved by radicals, but x⁵ - 6x + 3 cannot be solved by radicals.