MATH 141A: ARROW'S IMPOSSIBILITY THEOREM AND ULTRAFILTERS

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Arrow's theorem on voting systems says, roughly speaking, that any voting system which decides between three candidates or more must be imperfect (unless infinitely-many people are voting!). We make this precise and give a proof of this result using ultrafilters. The exposition here is primarily from the paper of Komjáth and Totik (*P. Komjáth and V. Totik. Ultrafilters, American Mathematical Monthly, 115(2008), 33–44*).

Roughly speaking, we are interested in studying an election (with three or more candidates), where each voter gives their preference (as a chain ranking the candidates). The question is then: how do we aggregate the preferences of everybody into a ranking of the candidates? More precisely:

Definition 1. For a set A, let $C(A)$ denote the set of all chains with universe A. A *voting system* consists of:

(1) A non-empty set V of voters.
(2) A set A of at least two candidates.
(3) A function $F : {}^V C(A) \to C(A)$ (sometimes called the *social welfare function*).

Intuitively, the social welfare function takes as input the rankings $(C_v)_{v \in V}$ of each voters and outputs an aggregate ranking. We call $(C_v)_{v \in V}$ a *preference profile* and the outcome $F((C_v)_{v \in V})$ an *election result*.

Definition 2. A voting system (V, A, F) is *perfect*[1] if:

- (Unanimity) For any $a, b \in A$, for any preference profile $(C_v)_{v \in V}$, if for all $v \in V$, $a <^{C_v} b$, then in the election result C also $a <^C b$. In other words, if b is preferred to a by every voter, then in the election result b is also preferred to a.
- (Independence of irrelevant alternatives) For any $a, b \in A$, for any two preference profiles $(C_v)_{v \in V}$ and $(D_v)_{v \in V}$, if for all $v \in V$, $a <^{C_v} b$ if and only if $a <^{D_v} b$, then $a <^C b$ if and only if $a <^D b$, where C and D are the election results for $(C_v)_{v \in V}$ and $(D_v)_{v \in V}$. In other words, introduction of other candidates than a or b should not influence their order in an election.
- (Non-dictatorship) There is no $w \in V$ such that for any $a, b \in A$, for any preferences profile $(C_v)_{v \in V}$ with election result C, $a <^{C_w} b$ implies $a <^C b$. In other words, there is no dictator w who decides the result of the election.

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[1]This is not standard terminology.

Perfect voting systems do exist in certain cases. First, if you have only two candidates to elect, there are no problems:

Theorem 3. If A and V are finite sets with $|A| = 2 \leq |V|$, there is a function F such that (V, A, F) is a perfect voting system.

Proof. Write $A = \{a, b\}$ and let $F((C_v)_{v \in V})$ rank b ahead of a if $|\{v \in V \mid a <^{C_v} b\}| \geq \frac{|V|}{2}$, and otherwise rank a ahead of b. This is clearly a non-dictatorial system also satisfying the unanimity property. As for the independence of irrelevant alternatives, it holds vacuously since there are no more than two candidates. $\qquad \square$

To see one issue when there are more than two candidates, consider the following two examples:

Example 4. Assume V is a non-empty finite set of voters. Assume (for simplicity), there are three candidates, a, b, and c, and $|V| \geq 10$.

(1) Suppose that we rank the candidates according to simple majority voting: given a preference profile $(C_v)_{v \in V}$, a candidate x, and $i \in \{1, 2, 3\}$, let n_x^i be the number of voters v such that x is the ith element in C_v (e.g. if $i = 3$, x is the first choice of voter v). Define $C = F((C_v)_{v \in V})$ by $x <^C y$ if $n_x^3 < n_y^3$, or $n_x^3 = n_y^3$ and $n_x^2 < n_y^2$ (or $x < y$ if $n_x^i = n_y^i$ for $i = 1, 2$, where we arbitrarily specify a way to break ties, say $a < b < c$). In other words, candidate y is preferred to candidate x if more people have put y as their top choices. Ties are broken by looking at who wanted y as their second choices. One can check that this is a voting system, and it satisfies unanimity and non-dictatorship. However, independence of irrelevant alternatives is problematic: suppose for example that $|V| = 5020$, 2501 voters v have C_v equal to $a < c < b$, 2500 voters v have C_v equal to $b < c < a$, and 19 voters v have C_v equal to $b < a < c$. Then according to F, b should win the election. Nevertheless more people prefer a to b (this is a recurring problem in US presidential elections). To see why independence of irrelevant alternatives fail: consider what happens if we define a new choice profile D_v which is like C_v, except that the 19 voters that preferred c in C_v have D_v equal to $c < b < a$, then a would win the election.

(2) In some countries like France, the presidential elections have two rounds. In the first, simple majority voting is used. The first two candidates are then elected in a second round of voting (try to describe the system using the formalism of these notes!). This system may mitigate the problem seen above in practice, but does not solve it in theory: in the election described above, almost everybody selected c as second choice, but nevertheless the two rounds system would have final outcome $c < b < a$. This translates to the following failure of independence of irrelevant alternatives: if the 2500 voters who preferred a had instead ranked $b < a < c$, then even though all those still prefer c to b, the election outcome would be $b < a < c$. In particular, the ordering of b and c in the final outcome was swapped while not changing for any individual voter (in this case, a was supposed to be the "irrelevant alternative").

We will see that these examples do not occur by accident (Corollary 11): with a finite number of voters and at least three candidates, there are *no* perfect voting systems whatsoever.

Interestingly, if there is an infinite number of voters, things are fine. First, recall:

Definition 5. A *filter* on a set I is a set F of subsets of I such that:

(1) $\emptyset \notin F$, $I \in F$.
(2) If $A \subseteq B \subseteq I$ and $A \in F$, then $B \in F$.
(3) If $A, B \in F$, then $A \cap B \in F$.

An *ultrafilter* on a set I is a filter U on I such that for any $X \subseteq I$, either $X \in U$ or $X^c = I - X \in U$. An ultrafilter U is called *principal* if $\{a\} \in U$ for some $a \in I$ or *nonprincipal* otherwise.

We have seen earlier in class that there are nonprincipal ultrafilters on any infinite sets (in fact, we proved that any filter extends to an ultrafilter). Using this, we can prove:

Theorem 6. If A and V are sets with $|A| \geq 2$ and V infinite, then there is a function F such that (V, A, F) is a perfect voting system.

Proof. Let U be a nonprincipal ultrafilter on V. Define $C = F((C_v)_{v \in V})$ as follows: for $a, b \in A$, $a <^C b$ if and only if $\{v \in V \mid a <^{C_v} b\} \in U$. Since U is not principal, there is no dictatorship in this voting system. Also, the unanimity property is true because $V \in U$. To see the independence of irrelevant alternatives, assume $(C_v)_{v \in V}$ and $(D_v)_{v \in V}$ are two preferences profiles, $a, b \in A$ and for all $v \in V$, $a <^{C_v} b$ if and only if $a <^{D_v} b$. Assume $a <^C b$. Then $\{v \in V \mid a <^{C_v} b\} \in U$, so by assumption $\{v \in V \mid a <^{D_v} b\} \in U$, and so $a <^D b$. Similarly, $a <^D b$ implies $a <^C b$, as desired. $\square$

For finite sets, the same proof cannot work:

Lemma 7. Assume U is an ultrafilter on a set I. If $A_1, \ldots, A_n$ are such that $A_1 \cup \ldots \cup A_n \in U$, then there exists $i \leq n$ such that $A_i \in U$. In particular, any ultrafilter on a finite set is principal.

Proof. First we prove that if $A \cup B \in U$, then $A \in U$ or $B \in U$. The result then follows by induction. Suppose for a contradiction that $A \notin U$ and $B \notin U$. Since U is an ultrafilter, $A^c \in U$ and $B^c \in U$. Then $A^c \cap B^c = (A \cup B)^c \in U$. However by assumption $A \cup B \in U$ so $\emptyset = (A \cup B) \cap (A \cup B)^c \in U$, a contradiction.

To see the "in particular" part, suppose that $I = \{a_1, \ldots, a_n\}$ is finite. Let $A_i = \{a_i\}$ for $i \leq n$. Then $A_1 \cup \ldots \cup A_n = I \in U$ by definition of a filter, so there exists $i \leq n$ such that $\{a_i\} \in U$, i.e. U is principal. $\square$

We will use the following converse, which is an assignment problem. For a set I, we call a non-empty collection P of subsets of I a *partition* of I if the sets in P are pairwise disjoint and their union is I (we allow the empty set to be in P).

Exercise 8. Assume I is a non-empty set and U is a collection of subsets of I. If $|P \cap U| = 1$ for any partition P of I with $|P| \leq 3$, then U is an ultrafilter on I.

The fact that an ultrafilter on a finite set is principal will translate to the fact that any voting system with finitely many voters and at least three candidates that satisfies unanimity and independence of irrelevant alternatives must be a dictatorship. In other words, in this case, there are no perfect voting systems.

To accomplish this translation, we will show that any voting system satisfying unanimity and independence of irrelevant alternatives is induced by an ultrafilter. The following notion will be key:

Definition 9. For a voting system (A, V, F), a subset X of V is called *decisive* if for any chain C with universe A and any preference profile $(C_v)_{v \in V}$, if $C_v = C$ for all $v \in X$, then $F((C_v)_{v \in V}) = C$. We say that $v \in V$ is *decisive* if $\{v\}$ is decisive (in other words, v is a dictator).

Intuitively, a set X of voters is decisive if the voters in X decide the election when they all vote the same way (i.e. have the same ranking). We will show:

Theorem 10 (Main theorem). If (A, V, F) is a voting system with $|A| \geq 3$ satisfying unanimity and independence of irrelevant alternatives, then the set of all decisive subsets of V is an ultrafilter.

Corollary 11 (Arrow's impossibility theorem). A voting system with a finite number of voters and at least three candidates to elect cannot be perfect.

Proof. Let A be a set with at least three elements and let V be a non-empty finite set. Assume that (A, V, F) is a voting system satisying unanimity and independence of irrelevant alternatives. By Theorem 10 (using that $|A| \geq 3$), the set U of all decisive subsets of V is an ultrafilter. By Lemma 7 (using that V is finite), U must be principal. In other words, there exists $v \in V$ such that $\{v\} \in U$. By definition of being decisive, v must be a dictator in this voting system. $\square$

The main theorem will be proven by proving increasingly stronger special cases. First, we introduce some notation. Fix a voting system (A, V, F) and distinct $v_1, v_2, v_3 \in V$. We will write (for example)

v_1:	$a_1 a_2 a_3 \dots a_n$
v_2:	$b_1 b_2 b_3 \dots b_m$
v_3:	$c_1 c_2 c_3 \dots c_k$
Outcome:	$d_1 d_2 d_3 \dots d_r$

If for *some* preference profile $(C_v)_{v \in V}$, if $C_{v_1} \models a_1 < a_2 \dots < a_n$, $C_{v_2} \models b_1 < b_2 < \dots < b_m$, and $C_{v_3} \models c_1 < \dots < c_k$, then $F(C_{v \in V}) \models d_1 < d_2 < \dots < d_r$. Note that, by the independence of irrelevant alternatives, if this holds for some preference profiles, this holds for *all* preference profiles satisfying the given orderings.

The following localized version of the definition of a decisive set will be useful:

Definition 12. For (A, V, F) a voting system and $a \neq b$ both in A, we say that a subset X of V is *decisive* for the pair (a, b) if for any chain C with universe A and any preference profile $(C_v)_{v \in V}$, if $a <^{C_v} b$ for all $v \in X$, then $a <^{F((C_v)_{v \in V})} b$. We say that $v \in V$ is *decisive* for (a, b) if $\{v\}$ is decisive for (a, b).

We prove some key properties of being decisive in case there are two voters:

Lemma 13. Assume (A, V, F) is a voting system with $|V| = 2$ satisfying unanimity and the independence of irrelevant alternatives. Assume $v \in V$ and a, b, c are distinct elements of A. If v is decisive for (a, b), then v is decisive for (a, c) and decisive for (b, c).

Proof. Write $V = \{v, w\}$. Since X is decisive for (a, b), we have:

v:	ab
w:	ba
Outcome:	ab

By unanimity (c is after b in both rankings):

v:	abc
w:	bca
Outcome:	abc

By the independence of irrelevant alternatives:

v:	ac
w:	ca
Outcome:	ac

This shows that v is decisive for (a, c). Now using unanimity (adding b before a in both ranking)

v:	bac
w:	cba
Outcome:	bac

Thus by the independence of irrelevant alternatives:

v:	bc
w:	cb
Outcome:	bc

This shows that v is decisive for (b, c). □

Lemma 14. Assume (A, V, F) is a voting system with $|V| = 2$, $|A| \geq 3$, satisfying unanimity and the independence of irrelevant alternatives. Let a, b be distinct elements of A. If v is decisive for (a, b), then v is decisive for (b, a).

Proof. Since $|A| \geq 3$, we can pick a candidate $c \neq a$, $c \neq b$. By Lemma 13, v is decisive for (b, c). Now using Lemma 13, where a, b, c there is b, c, a here, v is decisive for (b, a). □

We conclude that if there are only two voters we can obtain a dictator:

Lemma 15. If (A, V, F) is a voting system with $|V| = 2$, $|A| \geq 3$, satisfying unanimity and independence of irrelevant alternatives, then there is a decisive voter in V.

Proof. Write $V = \{v, w\}$. Using unanimity and independence of irrelevant alternatives, for each pair (a, b) of distinct candidates, either v or w is decisive for (a, b). So fix two distinct candidates a and b and assume without loss of generality that v is decisive for (a, b). Using Lemmas 13 and 14, we can swap any pair for which

a voter is decisive as well as change one of the components. Using this we see that v is decisive for (c, d) for any two distinct candidates c and d. Therefore v is decisive. □

Now we do this for three voters.

Lemma 16. If (A, V, F) is a voting system with $|V| = 3$, $|A| \geq 3$, satisfying unanimity and independence of irrelevant alternatives, then there is a decisive voter in V.

Proof. For any $v \in V$, we define a new voting system $S_v = (A, \{v, V \backslash \{v\}\}, F_v$, where $F_v(C_v, C_{V \backslash \{v\}}) = F(C_v, C_{V \backslash \{v\}}, C_{V \backslash \{v\}})$. It corresponds to the situation where the other two voters from V always vote in the same way. Now S_v has two voters, so by Lemma 15 it has a decisive voter. We consider two cases:

- There exists $v \in V$ such that in S_v the decisive voter is v. Then v will also be decisive in the original system. Indeed, suppose not. Then we must have a situation such as:

v:	ab
w:	ba
u:	ab
Outcome:	ba

  Now consider another new voting system $S_{v,ab} = (A, \{w, u\}, F_{v,ab})$, where $F_{v,ab}(C_w, C_u) = F(C, C_w, C_u)$, where C is any chain with universe A where $a < b$. By Lemma 15, $S_{v,ab}$ has a decisive voter. This decisive voter better be w. However since v was decisive in S_v, we must have:

v:	ab
w:	ba
u:	ba
Outcome:	ab

  This contradicts that w was decisive in $S_{v,ab}$.
- For every $v \in V$, the decisive voter in S_v is $V \backslash \{v\}$. So say $V = \{v, w, u\}$ and fix a pair (a, b) of distinct candidates. We must have:

v:	ab
w:	ba
u:	ba
Outcome:	ba

  as well as:

v:	ab
w:	ba
u:	ab
Outcome:	ab

  But now consider again the voting system $S_{v,ab}$ defined in the proof of the first case. As before, it is a two voter system so must have a decisive voter. If this voter is w, this contradicts the second table. If it is u, this contradicts the first table.

□

Now are now ready to prove the main theorem:

Proof of Theorem 10. Let U be the set of all decisive subsets of V. Observe that two decisive sets cannot be disjoint, so if P is a partition of V, at most one of the sets in the partition is in U. Assume in addition that $|P| \leq 3$. We show that one of the sets in the partition is in U. This will be enough by Exercise 8. Consider the voting system $S_P = (A, P, F_P)$, where $F_P((C_X)_{X \in P}) = F((C_v)_{v \in V})$, where $C_v = C_X$ if and only if $v \in X$. In other words, S_P is the voting system when each of the member of the same set of the partition rank candidates the same way. Now S_P has one, two or three voters so by Lemmas 15 or 16 there is $X \in P$ such that X is decisive in S_P (if P has one element, this is trivial). We claim that X is decisive also in the original voting system. Indeed, fix a chain C with universe A, and set $C_v = C$ for all $v \in X$. Let $(C_v)_{v \in V \setminus X}$ be possibly different chains. Fix $a <^C b$. Then we can partition $V \setminus X$ into $Y_1 \cup Y_2$, where $Y_1 = \{v \in V \setminus X \mid a <^{C_v} b\}$, $Y_2 = \{v \in V \setminus X \mid b <^{C_v} a\}$. Then $P' = (X, Y_1, Y_2)$ is another partition of V. In $S_{P'}$, there is again a decisive voter, and that voter better be X (because it is decisive when Y_1 and Y_2 vote the same way). Thus (by independence of irrelevant alternatives) in the final outcome, we must have a ranked below b. $\qquad\square$