

MATH 141A: THE COMPLETENESS THEOREM

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1. INTRODUCTION

In essence, the *completeness theorem* says that *being provable* is the same as being *a semantic consequence*. In symbols, if A is a set of axioms and ϕ is a sentence, $A \vdash \phi$ if and only if $A \models \phi$, where the symbol \vdash (read “proves”) means “there is a proof of ϕ using the axioms of A ” (it will be defined precisely later), and the symbol \models ($A \models \phi$ reads “ ϕ is a semantic consequence of A ”) was defined already: $A \models \phi$ means that any model of A is also a model of ϕ .

Another more vague (but better for cocktail parties) statement of the completeness theorem is: “something is true if and only if it is provable”.

The easy direction of the completeness theorem is to show that $A \vdash \phi$ implies $A \models \phi$: If we can prove ϕ from the axioms, presumably this will imply that any model of the axioms is a model of ϕ . The hard direction is to show that $A \models \phi$ implies $A \vdash \phi$. Even if we somehow manage to see that ϕ holds in every model of A , how do we know there is a proof of that fact?

An interesting consequence of the completeness theorem is the compactness theorem. Indeed, it will be almost immediate that if $A \vdash \phi$ then there is a finite subset A_0 of A such that $A_0 \vdash \phi$: regardless of what you mean by a proof, any reasonable definition should imply that a proof has a finite number of steps, and hence uses only a finite number of axioms. Thus if we start with a set A of axioms and we assume that it is inconsistent, then by definition we must have that $A \models \perp$ (where \perp is any sentence that is always false, such as $(\exists x)(x \neq x)$). By completeness, $A \vdash \perp$, but then this means there exists a finite subset A_0 of A such that $A_0 \vdash \perp$, and hence $A_0 \models \perp$. Thus any inconsistent set of axioms has a finite inconsistent subset. This is exactly the contrapositive of the compactness theorem!

2. PROVABILITY

We start by defining proofs carefully. There are many equivalent way of doing this. We try to choose one that is convenient to work with mathematically, although maybe not really convenient to use to write “actual proofs”. The treatment here is inspired from that in Kunen’s book on the foundations of mathematics. Throughout this section, we are fixing a signature σ .

First, we define the notion of a *propositional tautology*: it is simply a formula whose truth is apparent just from looking at the connectives “ \wedge , \vee , and \neg ”. For example, $\psi \vee \neg\psi$ is a propositional tautology for any formula ψ . More precisely:

Definition 2.1 (Propositional tautology).

- (1) A formula ϕ is *basic* if it is *not* of the form $\phi_1 \wedge \phi_2$, $\phi_1 \vee \phi_2$, or $\neg\psi$.
- (2) A *truth assignment* is a function v from the set of basic formulas into the set $\{0, 1\}$ (we think of 0 as meaning false, 1 as meaning true).
- (3) Given a truth assignment v , we define for any (not necessarily) formula ϕ its assignment $\bar{v}(\phi) \in \{0, 1\}$ by induction on the complexity of ϕ as follows:
 - If ϕ is basic, $\bar{v}(\phi) = v(\phi)$.
 - If ϕ is not basic and ϕ is $\neg\psi$, then $\bar{v}(\phi) = 1 - \bar{v}(\psi)$.
 - If ϕ is not basic and ϕ is $\phi_1 \wedge \phi_2$, then $\bar{v}(\phi) = \bar{v}(\phi_1) \cdot \bar{v}(\phi_2)$.
 - If ϕ is not basic and ϕ is $\phi_1 \vee \phi_2$, then $\bar{v}(\phi) = 1 - (1 - \bar{v}(\phi_1)) \cdot (1 - \bar{v}(\phi_2))$.
- (4) We call a formula ϕ a *propositional tautology* if $\bar{v}(\phi) = 1$ for any truth assignment v .

Example 2.2.

- For any formula ϕ , the formula $\phi \vee \neg\phi$ is a propositional tautology.
- For any formula ψ , the formula $\psi \leftrightarrow \psi$ is a propositional tautology (recall that, strictly speaking, $\psi \leftrightarrow \psi$ is only an abbreviation for a certain formula involving just negations, conjunctions, and disjunctions).
- The formula $(\forall x)(x = x)$ is not a propositional tautology (even though it is true in any model!) because it is a basic formula, and one can define a truth assignment v such that $v((\forall x)(x = x)) = 0$.

To deal with sentences, the following definition is also useful:

Definition 2.3. A *universal closure* of a formula ϕ is any *sentence* of the form $(\forall x_1)(\forall x_2) \dots (\forall x_n)\phi$, for $n < \omega$.

Definition 2.4. A *logical axiom* is a universal closure of a formula of one of the following types (below, ϕ, ψ denote arbitrary formulas the letters x, y, z denote arbitrary variables, and the letters s, t denote arbitrary terms):

- (1) Propositional tautologies.
- (2) $(\forall x)(\phi \rightarrow \psi) \rightarrow ((\forall x)(\phi) \rightarrow (\forall x)(\psi))$.
- (3) $\phi \rightarrow (\forall x)(\phi)$, where x is *not* a free variable of ϕ .
- (4) $(\forall x\phi(x)) \rightarrow \phi(t)$, where x is a free variable of ϕ , and $\phi(t)$ denotes the formula obtained by substituting t for x .
- (5) $\phi(t) \rightarrow (\exists x)(\phi(x))$, where x is a free variable of ϕ .
- (6) $(\forall x)(\neg\phi) \leftrightarrow \neg(\exists x)(\phi)$.
- (7) $x = x$.
- (8) $x = y \rightarrow y = x$.
- (9) $(x = y \wedge y = z) \rightarrow x = z$.
- (10) $(x_1 = y_1 \wedge \dots \wedge x_n = y_n) \rightarrow f(x_1, \dots, x_n) = f(y_1, \dots, y_n)$, where f is a function symbol of arity n .
- (11) $(x_1 = y_1 \wedge \dots \wedge x_n = y_n) \rightarrow (r(x_1, \dots, x_n) \leftrightarrow r(y_1, \dots, y_n))$, where r is a relation symbol of arity n .

Exercise 2.5. For any logical axiom ϕ , $\models \phi$. That is, for any σ -structure M , $M \models \phi$.

Definition 2.6. Assume that A is a set of sentences and ϕ is a sentence. A *formal proof of ϕ from A* is a finite sequence $\phi_1, \phi_2, \dots, \phi_n$ of sentences such that:

- (1) ϕ_n is ϕ .

- (2) For all $i \leq n$, one of the following is true:
- (a) $\phi_i \in A$.
 - (b) ϕ_i is a logical axiom.
 - (c) For some $j < i$ and $k < i$, ϕ_i follows from ϕ_j and ϕ_k by the law of “modus ponens”. In other words, ϕ_i is $\phi_j \rightarrow \phi_i$.

We write $A \vdash \phi$ (“ A proves ϕ ”) if there exists a formal proof of ϕ from A . We write $\psi \vdash \phi$ instead of $\{\psi\} \vdash \phi$, and $\vdash \phi$ instead of $\emptyset \vdash \phi$. Note that all this depends on σ , so when we want to make it clear we write $A \vdash_\sigma \phi$.

In words, a proof from A is a finite sequence of steps, where each step is either a logical axiom, an axiom from A , or something that can be inferred from the previous steps using modus ponens.

Example 2.7. Fix formulas ϕ and ψ . We show that $\{\phi \wedge \psi\} \vdash \phi$. A formal proof is given by ϕ_1, ϕ_2, ϕ_3 , where ϕ_1 is $\phi \wedge \psi$ (a member of $A = \{\phi \wedge \psi\}$), ϕ_2 is $(\phi \wedge \psi) \rightarrow \phi$ (a propositional tautology), and ϕ_3 is ϕ (obtained from ϕ_1 and ϕ_2 via modus ponens). Doing more complicated proofs with this system is quite cumbersome! Try proving that $(\exists x)(\neg\phi(x)) \leftrightarrow \neg(\forall x)\phi(x)$.

Example 2.8. Assume ϕ is a formula with free variable x . Assume that y is a variable not occurring anywhere in ϕ . Then $(\forall y)\phi(y) \vdash (\forall x)\phi$, where $\phi(y)$ denotes the formula ϕ where x has been replaced with y everywhere. A formal proof is given by:

- (1) $(\forall y)\phi(y)$ [given]
- (2) $(\forall x)(\forall y\phi(y) \rightarrow \phi(x))$ [logical axiom of type (4)]
- (3) $(\forall x)(\forall y\phi(y) \rightarrow \phi(x)) \rightarrow ((\forall x\forall y)\phi(y) \rightarrow (\forall x)\phi(x))$ [logical axiom of type (2)]
- (4) $(\forall x\forall y)\phi(y) \rightarrow (\forall x)\phi(x)$ [modus ponens from lines 2 and 3]
- (5) $(\forall y)\phi(y) \rightarrow (\forall x\forall y)\phi(y)$ [logical axiom of type (3)]
- (6) $(\forall x\forall y)\phi(y)$ [modus ponens from line 1 and 5]
- (7) $(\forall x)\phi(x)$ [modus ponens from line 4 and 6]

Remark 2.9. If $A \vdash \phi$, there is a finite set $A_0 \subseteq A$ such that $A_0 \vdash \phi$. Indeed, let ϕ_1, \dots, ϕ_n be a proof of ϕ from A . Then one can take $A_0 = \{\phi_1, \dots, \phi_n\} \cap A$.

We can already verify the easy direction of the completeness theorem.

Lemma 2.10 (Soundness). Assume A is a set of sentences and ϕ is a sentence. If $A \vdash \phi$, then $A \models \phi$.

Proof. Let ϕ_1, \dots, ϕ_n be a proof of ϕ from A . We prove by induction on i that $A \models \phi_i$. If $i = 1$, then ϕ_1 is either in A (in which case clearly $A \models \phi_1$), or a logical axiom (in which case $A \models \phi_1$ by Exercise 2.5).

Assume now that $i > 1$ and $A \models \phi_j$ for all $j < i$. Again, if $\phi_i \in A$ or ϕ_i is a logical axiom, we have that $A \models \phi_i$. If for some $j, k < i$, ϕ_i follows from ϕ_j and ϕ_k by modus ponens, then using modus ponens in the “real world”, together with the fact that $A \models \phi_j$ and $A \models \phi_k$ (given by the induction hypothesis), we easily get that $A \models \phi_i$. \square

We now prove a few basic properties of provability:

Lemma 2.11 (Transitivity). Assume A is a set of sentences and $\phi_1, \dots, \phi_n, \psi$ are sentences. If for all $i \leq n$, $A \vdash \phi_i$ and $A \cup \{\phi_1, \dots, \phi_n\} \vdash \psi$, then $A \vdash \psi$.

Proof. Let $\phi_i^1, \dots, \phi_i^{m_i}$ be a proof of ϕ_i from A , and let ψ_1, \dots, ψ_m be a proof of ψ from $A \cup \{\phi_1, \dots, \phi_n\}$. Then the concatenation: $\phi_1^1, \dots, \phi_1^{m_1}, \dots, \phi_n^1, \dots, \phi_n^{m_n}, \psi_1, \dots, \psi_m$ is a proof of ψ from A . \square

Theorem 2.12 (The deduction theorem). Assume A is a set of sentences, and ϕ, ψ are sentences. Then $A \cup \{\phi\} \vdash \psi$ if and only if $A \vdash \phi \rightarrow \psi$.

Proof. First assume that $A \vdash \phi \rightarrow \psi$. Assume ψ_1, \dots, ψ_n is a formal proof witnessing it. Then $\psi_1, \dots, \psi_n, \phi, \psi$ is a formal proof of ψ from $A \cup \{\phi\}$. Indeed, $\phi \in A \cup \{\phi\}$ and ψ follows from ϕ and ψ_n by the rule of modus ponens.

For the converse, assume that $A \cup \{\phi\} \vdash \psi$. Let $\psi_1, \psi_2, \dots, \psi_n$ be the corresponding formal proof. We prove by induction on $i \leq n$ that $A \vdash \phi \rightarrow \psi_i$. Fix i and assume inductively that this holds for all $j < i$. There are several cases:

- If ψ_i is in A , or ψ_i is a logical axiom, then:
 - (1) ψ_i [in A /logical axiom]
 - (2) $\psi_i \rightarrow (\phi \rightarrow \psi_i)$ [propositional tautology]
 - (3) $\phi \rightarrow \psi_i$ [modus ponens from the first and second lines]
 gives a formal proof of $\phi \rightarrow \psi_i$ from A .
- If ψ_i is ϕ , then $\phi \rightarrow \psi_i$ is a propositional tautology, so $A \vdash \phi \rightarrow \psi_i$.
- If ψ_i is obtained from ψ_j and ψ_k by modus ponens, $j, k < i$, and ψ_k is $\psi_j \rightarrow \psi_i$, then we can construct a proof of $\phi \rightarrow \psi_i$ as follows:
 - (1) $A \vdash \phi \rightarrow \psi_j$ [induction hypothesis]
 - (2) $A \vdash \phi \rightarrow (\psi_j \rightarrow \psi_i)$ [induction hypothesis]
 - (3) $A \vdash (\phi \rightarrow \psi_j) \rightarrow ((\phi \rightarrow (\psi_j \rightarrow \psi_i)) \rightarrow (\phi \rightarrow \psi_i))$ [propositional tautology]
 - (4) $A \vdash ((\phi \rightarrow (\psi_j \rightarrow \psi_i)) \rightarrow (\phi \rightarrow \psi_i))$ [modus ponens from lines 1 and 3]
 - (5) $A \vdash \phi \rightarrow \psi_i$ [modus ponens from lines 2 and 4]

\square

We now examine the definition of inconsistency for proofs.

Definition 2.13. A set A of sentences is called *syntactically inconsistent* if there is a sentence ϕ so that $A \vdash \phi$ and $A \vdash \neg\phi$. If A is not syntactically inconsistent, we say that it is *syntactically consistent*.

We give a few equivalent definitions:

Definition 2.14. Let T be the sentence $(\forall x)(x = x)$. Let \perp be the sentence $\neg T$.

Lemma 2.15. The following are equivalent for a set A of sentences:

- (1) A is syntactically inconsistent.
- (2) $A \vdash \psi$ for any sentence ψ .
- (3) $A \vdash \perp$.

Proof.

- (2) implies (3): Trivial.
- (3) implies (1): We have that $A \vdash T$ (because T is a logical axiom of type (7)). Therefore if $A \vdash \perp$, we can take ϕ to be T in the definition of syntactic inconsistency.
- (1) implies (2): Assume that $A \vdash \phi$ and $A \vdash \neg\phi$. Fix a sentence ψ . We can construct a formal proof of ψ from A as follows:
 - $A \vdash \phi$ [assumption]
 - $A \vdash \neg\phi$ [assumption]
 - $A \vdash \phi \rightarrow (\neg\phi \rightarrow \psi)$ [propositional tautology]
 - $A \vdash \psi$ [modus ponens used twice with the previous lines]

□

Lemma 2.16 (Proof by contradiction). Assume A is a set of sentences and ϕ is a sentence.

- (1) $A \vdash \phi$ if and only if $A \cup \{\neg\phi\}$ is syntactically inconsistent.
- (2) $A \vdash \neg\phi$ if and only if $A \cup \{\phi\}$ is syntactically inconsistent.

Proof. We prove the first statement, and leave the second one as an exercise. Assume first that $A \vdash \phi$. Then the same formal proof shows that $A \cup \{\neg\phi\} \vdash \phi$, but also $A \cup \{\neg\phi\} \vdash \neg\phi$, so ϕ witnesses that $A \cup \{\phi\}$ is syntactically inconsistent.

Conversely, assume that $A \cup \{\neg\phi\}$ is syntactically inconsistent. By Lemma 2.15, $A \cup \{\neg\phi\}$ proves anything and in particular it proves ϕ . By the deduction theorem, $A \vdash \neg\phi \rightarrow \phi$. Furthermore, $(\neg\phi \rightarrow \phi) \rightarrow \phi$ is a propositional tautology, so using modus ponens we obtain a formal proof of ϕ from A . □

To deal with quantifiers, we prove for each quantifier an *introduction rule* (telling us how to deduce a statement involving the quantifier from a statement without the quantifier) and an *elimination rule* (telling us how to remove the quantifier from a statement).

Theorem 2.17. Assume A is a set of sentences in the language of σ , ϕ is a formula in the language of σ with free variable x , ψ is a sentence in the language of σ , and $\sigma' = \sigma \cup \{c\}$, where c is a constant symbol not in σ .

- (Introduction rule for \forall): If $A \vdash_{\sigma'} \phi(c)$, then $A \vdash_{\sigma} (\forall x)\phi$.
- (Elimination rule for \forall): If $A \vdash_{\sigma} (\forall x)\phi$, then $A \vdash_{\sigma} \phi(t)$ for any term t .
- (Introduction rule for \exists): If for some term t , $A \vdash_{\sigma} \phi(t)$, then $A \vdash_{\sigma} (\exists x)\phi$.
- (Elimination rule for \exists): If $A \vdash_{\sigma} (\exists x)\phi$ and $A \cup \{\phi(c)\} \vdash_{\sigma'} \psi$, then $A \vdash_{\sigma} \psi$.

It should be clear why, intuitively, the elimination rule for \forall and the introduction rule for \exists are true. The introduction rule for \forall says that if we can prove a statement $\phi(c)$ for an “arbitrary” constant symbol c , then we can prove $(\forall x)\phi$. Similarly, the elimination rule for \exists says that if we know that $(\exists x)\phi$, and from assuming $\phi(c)$ we can deduce ψ , then we can deduce ψ directly.

Proof. The elimination rule for \forall is easy to prove: fix a term t . Then $A \vdash_{\sigma} (\forall x)\phi \rightarrow \phi(t)$ (it is a logical axiom of type (4)), so if $A \vdash_{\sigma} (\forall x)\phi$, then using modus ponens we also get that $A \vdash_{\sigma} \phi(t)$. The introduction rule for \exists can be proven similarly, using the corresponding logical axiom of type (5).

Before proving the introduction rule for \forall , we show how it implies the elimination rule for \exists . Assume that $A \vdash_{\sigma} (\exists x)\phi$ and $A \cup \{\phi(c)\} \vdash_{\sigma'} \psi$. We want to see that $A \vdash_{\sigma} \psi$. We show that $A \cup \{(\exists x)\phi\} \vdash_{\sigma} \psi$, which is enough by transitivity (Lemma 2.11). Since $A \cup \{\phi(c)\} \vdash_{\sigma'} \psi$, proof by contradiction (Lemma 2.16) implies that $A \cup \{\phi(c)\} \cup \{\neg\psi\}$ is syntactically inconsistent, and therefore (by another application of proof by contradiction) $A \cup \{\neg\psi\} \vdash_{\sigma'} \neg\phi(c)$. By the introduction rule for \forall , $A \cup \{\neg\psi\} \vdash_{\sigma} (\forall x)(\neg\phi)$. We also have that $A \cup \{\neg\psi\} \vdash_{\sigma} (\forall x)(\neg\phi) \rightarrow \neg(\exists x)\phi$ (this follows from modus ponens using a logical axiom of type (6) and the propositional tautology $(\psi_1 \leftrightarrow \psi_2) \rightarrow (\psi_1 \rightarrow \psi_2)$). Therefore using modus ponens, $A \cup \{\neg\psi\} \vdash_{\sigma} \neg(\exists x)\phi$. Using proof by contradiction again (Lemma 2.16), $A \cup \{\neg\psi\} \cup \{(\exists x)\phi\}$ is syntactically inconsistent, and therefore (by another appeal to proof by contradiction) $A \cup \{(\exists x)\phi\} \vdash_{\sigma} \psi$.

Finally, we prove the introduction rule for \forall . Assume that $A \vdash_{\sigma'} \phi(c)$. Let $\phi_1, \phi_2, \dots, \phi_n$ be a formal proof of $\phi(c)$ from A . Let y be a variable not appearing anywhere in the proof (note that x may appear in other ϕ_i 's than ϕ , in unrelated ways). For $i \leq n$, let $\phi_i(y)$ denote ϕ_i where c is replaced by y everywhere. We prove by induction on i that $A \vdash_{\sigma} (\forall y)\phi_i(y)$. This is enough: then $A \vdash_{\sigma} (\forall y)\phi(y)$, so using Example 2.8 and transitivity (Lemma 2.11), $A \vdash_{\sigma} (\forall x)\phi$, as desired.

Let us fix i and assume inductively that $A \vdash_{\sigma} (\forall y)\phi_j$ for all $j < i$. There are several cases:

- ϕ_i is a logical axiom. Then we can check the logical axioms of each type to see that $(\forall y)\phi_i(y)$ is also a logical axiom (of the same type). In essence, this is because it does not matter whether we use the constant symbol c or the variable y (which crucially does not appear in ϕ_i).
- ϕ_i is in A . Then ϕ_i is in the language of σ (c does not appear anywhere), so using a logical axiom of type (3) and modus ponens, $A \vdash_{\sigma} (\forall y)\phi_i(y)$.
- ϕ_i follows from ϕ_j and ϕ_k by modus ponens for $j, k < i$, so ϕ_k is $\phi_j \rightarrow \phi_i$. Then we construct a proof that $A \vdash_{\sigma} (\forall y)\phi_i(y)$ as follows:
 - (1) $A \vdash_{\sigma} (\forall y)\phi_j(y)$ [induction hypothesis]
 - (2) $A \vdash_{\sigma} (\forall y)(\phi_j(y) \rightarrow \phi_i(y))$ [induction hypothesis for ϕ_k]
 - (3) $A \vdash_{\sigma} (\forall y)(\phi_j(y) \rightarrow \phi_i(y)) \rightarrow ((\forall y)\phi_j(y) \rightarrow (\forall y)\phi_i(y))$ [logical axiom of type (2)]
 - (4) $A \vdash_{\sigma} (\forall y)(\phi_j(y) \rightarrow (\forall y)\phi_i(y))$ [modus ponens with lines 2 and 3]
 - (5) $A \vdash_{\sigma} (\forall y)\phi_i(y)$ [modus ponens with line 1 and 4]

□

3. THE MODEL EXISTENCE THEOREM

We now have enough tools to prove the completeness theorem. The main theorem will show that any syntactically consistent set of sentences has a model.

Theorem 3.1 (The model existence theorem). *If A is a syntactically consistent set of sentences, then it has a model.*

Before starting the proof of the model existence theorem, let us see why it implies the completeness theorem.

Corollary 3.2 (The completeness theorem). Assume A is a set of sentences and ϕ is a sentence. Then $A \vdash \phi$ if and only if $A \models \phi$.

Proof. If $A \vdash \phi$, we have seen that $A \models \phi$ in Lemma 2.10. We prove the contrapositive of the converse: if $A \not\vdash \phi$, then $A \not\models \phi$. Assume that $A \not\vdash \phi$. By Lemma 2.16 (“proof by contradiction”), this means that $A \cup \{\neg\phi\}$ is syntactically consistent. By the model existence theorem, $A \cup \{\neg\phi\}$ has a model M . Then $M \models A$ but $M \not\models \phi$, so M witnesses that $A \not\models \phi$. \square

Note that, unfortunately, the proof of the completeness theorem is not constructive, in the sense that it does not tell us what the proof of ϕ from A should be. An easy consequence of completeness is compactness:

Corollary 3.3 (The compactness theorem). Assume A is a set of sentences. If every finite subset of A has a model, then A has a model.

Proof. Assume that A does not have a model. Then $A \models \perp$, so by completeness $A \vdash \perp$. By Remark 2.9, there exists a finite subset $A_0 \subseteq A$ such that $A_0 \vdash \perp$. By completeness again, $A_0 \models \perp$. This shows that A_0 does not have a model, a contradiction. \square

Now let us work toward the proof of the model existence theorem. The proof has three steps: first, we will add *witnessing terms* to make sure that sentences in A of the form $(\exists x)\phi$ have some closed terms witnessing their truth. Next, we will enlarge the set A as much as possible to make it complete. Finally, we will show that the set of sentences thus obtained has a canonical and natural model. We begin with definitions to make all this precise:

Definition 3.4. A set A of sentences is said to have *witnessing terms* if for any formula ϕ with a free variable x , if $(\exists x)\phi \in A$, then there exists a closed term t (i.e. a term without free variables) such that $\phi(t) \in A$. We call t the *a witnessing term* for $(\exists x)\phi$.

The next lemma says that adding a witnessing term preserves consistency.

Lemma 3.5 (Adding one witnessing constant). Assume A is a syntactically consistent set of sentences in the language of σ . Assume c is a constant symbol not in σ , ϕ is a formula with a free variable x , and $(\exists x)\phi \in A$. Then $A \cup \{\phi(c)\}$ is syntactically consistent (in the language of $\sigma \cup \{c\}$).

Proof. Assume for a contradiction that $A \cup \{\phi(c)\}$ is syntactically inconsistent. Then by Lemma 2.16, $A \vdash \neg\phi(c)$. By the introduction rule for \forall (Theorem 2.17), $A \vdash (\forall x)\neg\phi$. Using a logical axiom of type (6), the propositional tautology $(\psi_1 \leftrightarrow \psi_2) \rightarrow (\psi_1 \rightarrow \psi_2)$, and two applications of modus ponens, we deduce that $A \vdash \neg(\exists x)\phi$. Since we assumed that $(\exists x)\phi \in A$, $A \vdash (\exists x)\phi$. Therefore A is syntactically inconsistent, a contradiction. \square

Lemma 3.5 can be iterated: the following consequence of the finite character of proofs will get us through the limit stages:

Lemma 3.6. Assume α is a limit ordinal and $(A_i)_{i < \alpha}$ is an increasing chain of syntactically consistent set of sentences (that is, $A_i \subseteq A_j$ for all $i < j < \alpha$). Then $\bigcup_{i < \alpha} A_i$ is syntactically consistent.

Proof. Let $A = \bigcup_{i < \alpha} A_i$. If A is syntactically inconsistent, then (Lemma 2.15) $A \vdash \perp$. By Remark 2.9, there exists a finite $B \subseteq A$ such that $B \vdash \perp$. Since α was a limit ordinal, this means that $B \subseteq A_i$ for some $i < \alpha$, so A_i is itself syntactically inconsistent, a contradiction. \square

Lemma 3.7. If A is syntactically consistent, there exists a syntactically consistent set B (in an expanded signature) such that $A \subseteq B$ and every existential sentence in A has witnessing terms.

Proof. Let $(\phi_i)_{i < \lambda}$ enumerate all the existential sentences in A . Build $(A_i)_{i < \lambda}$ syntactically consistent such that $A_0 = A$, $A_i \subseteq A_j$ for $i < j < \lambda$, and for all $i < \lambda$, ϕ_i has a witnessing term in A_{i+1} . The successor stage can be implemented using Lemma 3.5 and the limit stage using Lemma 3.6. \square

We could iterate the last lemma ω -times to obtain a syntactically consistent set with witnessing terms, however we will also want to require completeness:

Definition 3.8. We say a set A of sentences in the language of σ is *maximal* (in the language of σ) if for any sentence ϕ in the language of σ , either $\phi \in A$ or $\neg\phi \in A$.

As for witnessing terms, we can increase any set of sentences to be maximal, while preserving consistency.

Lemma 3.9. Assume A is syntactically consistent and ϕ is a sentence. Then either $A \cup \{\phi\}$ is syntactically consistent or $A \cup \{\neg\phi\}$ is syntactically consistent.

Proof. If $A \cup \{\phi\}$ is syntactically inconsistent, then by Lemma 2.16, $A \vdash \neg\phi$. Thus if $A \cup \{\neg\phi\}$ was also syntactically inconsistent, we could use transitivity (Lemma 2.11) to deduce that A is syntactically inconsistent. \square

Lemma 3.10. If A is a syntactically consistent set of sentences in the language of σ , then A can be extended to a *maximal* syntactically consistent set of sentences in the language of σ .

Proof. Iterate Lemma 3.9 over all sentences in the language of σ , using Lemma 3.6 at limits. \square

We can add witnessing constant and extend to maximal sets at the same time to obtain:

Theorem 3.11. If A is a syntactically consistent set of sentences, then there exists a maximal syntactically consistent extension of A (in an extended signature) which has witnessing terms.

Proof. We build $(A_n)_{n < \omega}$ syntactically consistent such that $A_0 = A$, and for all $n < \omega$:

- $A_n \subseteq A_{n+1}$

- A_{2n+1} is maximal.
- In A_{2n+2} , any existential sentence from A_{2n+1} has a witnessing term.

This is possible by using Lemmas 3.10 and 3.7. At the end, let $B = \bigcup_{n < \omega} B_n$. This is syntactically consistent by Lemma 3.6. Moreover, it is easy to check from the construction that it is maximal and has witnessing terms. \square

We now show how to build a model for a maximal syntactically consistent set with witnesses:

Lemma 3.12. If A is a maximal syntactically consistent set of sentences in the language of σ and A has witnessing terms, then A has a model.

Proof. First, if the sentence $\neg(\exists x)(x = x)$ is in A , then the empty structure will be a model for A , so assume now that $(\exists x)(x = x)$ is in A (one of these cases must be true by maximality). Since A has witnessing terms, this means in particular that there is at least one closed term.

Let X be the set of all closed terms in the language of σ . Define an equivalence relation \sim on X by $t \sim s$ if and only if $t = s \in A$. This is an equivalence relation (you will prove this in assignment 8 – this is where the logical axioms regarding equality are used). Now let E be the set of \sim -equivalence classes. We define a σ -structure M as follows:

- The universe of M is the set E .
- If c is a constant symbol, let $c^M = [c]$.
- If f is a function symbol of arity n , let $f^M([t_1], [t_2], \dots, [t_n]) = [f(t_1, \dots, t_n)]$. Note that this is well-defined – it does not depend on the choice of the representatives for the equivalence classes (assignment 8).
- If r is a relation symbol of arity n , define $([t_1], \dots, [t_n]) \in r^M$ to be true if and only if $A \vdash r(t_1, \dots, t_n)$. Again, one can check that this is well-defined.

It remains to see that M is a model of A . First, an easy induction on term complexity shows that for any term $\tau(x_1, \dots, x_n)$ and any closed terms t_1, \dots, t_n , $\tau^M([t_1], \dots, [t_n]) = [\tau(t_1, \dots, t_n)]$ (the base cases are given by the interpretation of the constant symbols in M and the definition of the universe of M ; the inductive step is given by the interpretation of the function symbols in M). We will use this several times.

We prove something nominally stronger than the fact that M is a model of A . Fix a formula $\phi(x_1, \dots, x_n)$ in the language of σ . We show by induction on the complexity of ϕ that for any closed terms t_1, \dots, t_n $M \models \phi([t_1], \dots, [t_n])$ if and only if $\phi(t_1, \dots, t_n) \in A$. As usual, we assume without loss of generality that ϕ does not contain \vee or \forall .

- If ϕ is $r(s_1, \dots, s_m)$ for s_1, \dots, s_m (not necessarily closed) terms, then this is immediate from the definition of r^M and the claim about closed terms mentioned above.
- If ϕ is $t = s$ for t and s terms, then again the result is immediate from the definition of E : let t' and s' be the result of replacing the variables in t and s by t_1, \dots, t_n . Then $t' = s'$ is in A if and only if $t' \sim s'$ if and only if

$[t'] = [s']$ if and only if $(t')^M = (s')^M$. The last if and only if holds because of the claim on terms mentioned above.

- If ϕ is $\psi_1 \wedge \psi_2$, then (dropping t_1, \dots, t_n for notational clarity) $\psi_1 \wedge \psi_2 \in A$ if and only if $\psi_1 \in A$ and $\psi_2 \in A$ (assignment 8) if and only if $M \models \psi_1$ and $M \models \psi_2$ (induction hypothesis) if and only if $M \models \psi_1 \wedge \psi_2$.
- If ϕ is $\neg\psi$, then (again dropping t_1, \dots, t_n for notational clarity) $\phi \in A$ if and only if $\psi \notin A$ (the left to right direction is by syntactical consistency of A , the right to left direction is by maximality) if and only if $M \not\models \psi$ (induction hypothesis) if and only if $M \models \phi$.
- If ϕ is $(\exists x)\psi(x, x_1, \dots, x_n)$, assume first that $M \models (\exists x)\psi(x, [t_1], \dots, [t_n])$. Then there exists b in E such that $M \models \psi(b, [t_1], \dots, [t_n])$. By definition of M , $b = [t]$ for some closed term t . By the induction hypothesis, $\psi(t, t_1, \dots, t_n) \in A$. Therefore by the introduction rule for \exists (Theorem 2.17), $\phi(t_1, \dots, t_n) \in A$.

Assume now that $\phi(t_1, \dots, t_n) \in A$. Then since A has witnessing terms, there exists a term t such that $\psi(t, t_1, \dots, t_n) \in A$. By the induction hypothesis, $M \models \psi([t], [t_1], \dots, [t_n])$. Therefore $M \models \phi([t_1], \dots, [t_n])$.

□

We conclude with the proof of the model existence theorem:

Proof of the model existence theorem. Assume A is a syntactically consistent set of sentences in the language of σ . By Theorem 3.11, there exists B extending A which is maximal syntactically consistent and has witnessing terms. By Lemma 3.12, B has a model M . Since $A \subseteq B$ the reduct $M \upharpoonright \sigma$ of M to the signature σ is the desired model of A . □