MATH 141A - MATHEMATICAL LOGIC I, FALL 2018
ASSIGNMENT 6

Due Monday, October 22 at the beginning of class (please submit your assignment on Canvas). Make sure to include your full name *and the list of your collaborators* (if any) with your assignment. You may discuss problems with others, but you may *not* keep a written record of your discussions. Please refer to the syllabus for details.

With regards to answering these problems, imagine that you are writing an answer to teach someone else in the class how to do the problem. In particular, you must give a complete outline for how you arrived at your answer. It is not sufficient to simply state a number or formula without providing the steps and reasoning that you used to produce the answer.

(1) Let M and N be σ-structures. Assume that s is an isomorphism from M to N, let $a_1, a_2, \ldots, a_n$ be members of in the universe of M, and let $\phi(x_1, \ldots, x_n)$ be a formula. Prove that $M \models \phi(a_1, \ldots, a_n)$ if and only if $N \models \phi(s(a_1), \ldots, s(a_n))$.

(2) For a fixed σ-structure M, and a number $n < \omega$, a set X of n-tuples in the universe of M is called *definable* if there exists a formula $\phi(x_1, \ldots, x_n)$ (in the language of σ) such that for any n-tuple $\bar{a}$ from M, $M \models \phi(\bar{a})$ if and only if $\bar{a} \in X$.

- (a) Explain why any infinite structure of cardinality strictly larger than $|\sigma|$ will have non-definable subsets.
- (b) Assume M is a σ-structure with universe E and let $n < \omega$. Show that $\emptyset$ and E^n are always definable.
- (c) Show that definable sets are closed under complements, finite unions, and finite intersections.
- (d) Consider the structure $M = (\mathbb{R}, +, \cdot)$. Show that the sets $\{0\}$ and $\{1\}$ are definable in M. Further show that the set $\{(a, b) \in \mathbb{R} \times \mathbb{R} \mid a < b\}$ is also definable.
- (e) Show that $\mathbb{N}$ is definable in $(\mathbb{Z}, +, \cdot)$, but not in $(\mathbb{Q}, <)$. *Hint: for the first part, you may want to use the following theorem from number theory: any natural number can be written as the sum of four squares.*

(3) (a) Prove that the following are equivalent for a chain $(C, <)$:

- (i) C is *not* a well-ordering.
- (ii) There exists a sequence $(a_n)_{n<\omega}$ such that $a_{n+1} < a_n$ for all $n < \omega$.

(b) Assume C is an infinite well-ordering. Show that there exists a chain D which is elementary equivalent to C but is not a well-ordering. *Hint: add constant symbols and use the compactness theorem.*

(4) Assume P is a (possibly infinite!) set of primes. Show that there exists a structure M such that M is elementarily equivalent to $(\mathbb{N}, +, \cdot, 0, 1)$ but

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there is a in the universe of M that is divisible exactly by the primes in P. That is, for any prime p, $p \in P$ if and only if

$$M \models (\exists y)(a = (\underbrace{1 + 1 + \ldots + 1}_{p \text{ times}})y)$$

Hint: use the compactness theorem.

(5) Let F be a filter on ω. Let $(a_n)_{n<\omega}$ be a sequence of real numbers and let a be a real number. We say that $(a_n)_{n<\omega}$ *F-converges to a* if for any $\epsilon > 0$, $\{n < \omega \mid |a_n - a| < \epsilon\} \in F$. We call a an *F-limit* of $(a_n)_{n<\omega}$.

(a) Show that if $(a_n)_{n<\omega}$ F-converges to both a and b, then $a = b$. *Note: this shows that the F-limit is unique if it exists.*

(b) Describe the F-limit if F is the Fréchet filter. What if F is a principal ultrafilter?

(c) Assume now that U is an ultrafilter. Show that any bounded sequence U-converges.