MATH 141A - MATHEMATICAL LOGIC I, FALL 2018
ASSIGNMENT 7

Due Friday, October 26 at the beginning of class (please submit your assignment on Canvas). Make sure to include your full name *and the list of your collaborators* (if any) with your assignment. You may discuss problems with others, but you may *not* keep a written record of your discussions. Please refer to the syllabus for details.

With regards to answering these problems, imagine that you are writing an answer to teach someone else in the class how to do the problem. In particular, you must give a complete outline for how you arrived at your answer. It is not sufficient to simply state a number or formula without providing the steps and reasoning that you used to produce the answer.

(1) Recall that we call a class K of σ-structures *elementary* if there exists a set A of sentences in the language of σ such that K is exactly the class of models of A. We call K *finitely axiomatizable* if we can take A finite. For example, the class of all infinite sets (in the empty signature) is elementary and the class of chains is finitely axiomatizable (chains can be described by three sentences: irreflexivity, transitivity, and totality).
   (a) Show that if K is finitely axiomatizable, then its complement (the class of σ-structures not in K) is also finitely axiomatizable.
   (b) Prove that the class of all *finite* sets (in the empty signature) is *not* elementary. *Hint: use the compactness theorem just like in the proof of the upward Löwenheim-Skolem theorem.*
   (c) Deduce that the class of all infinite sets (in the empty signature) is *not* finitely axiomatizable.

(2) Show that the interpretation of each function symbol is well-defined in the definition of the ultraproduct. That is, assume that U is an ultrafilter on a set I and $(M_i)_{i\in I}$ is a sequence of non-empty σ-structures. Let f be a function symbol of arity n, $(a_i^1)_{i\in I}$, $(b_i^1)_{i\in I}$, $(a_i^2)_{i\in I}$, $(b_i^2)_{i\in I}$ ..., $(a_i^n)_{i\in I}, (b_i^n)_{i\in I}$ be elements of $\prod_{i\in I} \text{univ}(M_i)$. Assume that $(a_i^k)_{i\in I} \sim (b_i^k)_{i\in I}$ for each $k \leq n$. Show that $(f^{M_i}(a_i^1, a_i^2, \ldots, a_i^n))_{i\in I} \sim (f^{M_i}(b_i^1, b_i^2, \ldots, b_i^n))_{i\in I}$. *(Recall that $\sim$ means equality modulo U: $(c_i)_{i\in I} \sim (d_i)_{i\in I}$ if and only if $\{i \in I \mid c_i = d_i\} \in U$).*

(3) First a general definition. Let U be an ultrafilter on a set I, and let M be a non-empty σ-structure. For $i \in I$, let $M_i = M$. We write M^U for $\prod_{i\in I} M_i/U$. This is called the *ultrapower* of M by U.

   For the rest of this problem, set $M := (\mathbb{N}, +, \cdot, <, 0, 1)$, and let U be a nonprincipal ultrafilter on $I = \omega$.
   (a) Fix $m < \omega$. Give a function $x : \omega \to \omega$ such that $M^U \models [x] = \underbrace{1 + 1 + \ldots + 1}_{m \text{ times}}$.

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(b) For $a, b \in M^U$, let us say that b is *well above* a if for any $m < \omega$, $M^U \models a + \underbrace{1 + 1 + \ldots + 1}_{m \text{ times}} < b$. Show that [id] is well above 0^{M^U}, where id is the function: $\text{id} : \omega \to \omega$ given by $\text{id}(n) = n$.
(c) Show that for any $a \in M^U$, there exists $b \in M^U$ which is well above a.
(d) Show that for any $a \in M^U$ that is well above 0^{M^U}, there exists $b \in M^U$ such that a is well above b and b is well above 0^{M^U}.
(e) Show that for any (possibly infinite) set P of primes there exists $a \in M^U$ that is divisible exactly by the primes in P.

(4) An *elementary embedding* from a σ-structure M to a σ-structure N is an injection $s : M \to N$ such that s is an isomorphism onto $s[M]$ and $s[M] \preceq N$ *(note: we are abusing notation here, and should have really written $s : univ(M) \to univ(N)$. We also have written $s[M]$ for the unique substructure of N with universe $s[univ(M)]$).*
(a) Prove that the following are equivalent, for a function $s : M \to N$.
(i) s is an elementary embedding.
(ii) For any formula $\phi(x_1, \ldots, x_n)$ and any $a_1, \ldots, a_n$ in the universe of M, $M \models \phi(a_1, \ldots, a_n)$ if and only if $M \models \phi(s(a_1), \ldots, s(a_n))$.
(b) Assume that $s : M \to N$ is an elementary embedding. Prove that there exists a σ-structure M' and an isomorphism $t : M' \cong N$ such that $M \preceq M'$ and t extends s. *Hint: first think of the structures as chains and try to draw a picture of the situation.*
(c) Let U be an ultrafilter on some set I and let M be a non-empty σ-structure. Prove that the function $s : M \to M^U$ given by $s(a) = [c_a]$ (where $c_a : I \to M$ is the constantly a function: $c_a(i) = a$) is an elementary embedding. It is called the *canonical embedding* of M into M^U.
(d) Assume now that U is nonprincipal and $I = \text{univ}(M)$. Prove that the canonical embedding is proper (that is, it is not surjective). Deduce that any infinite σ-structure has a proper elementary extension.

(5) (Extra credit) Let N be countable, elementarily equivalent but not isomorphic to $(\mathbb{N}, +, \cdot, <, 0, 1)$. Describe the chain $(\text{univ}(N), <^N)$ up to isomorphism.