MATH 141A - MATHEMATICAL LOGIC I, FALL 2018
ASSIGNMENT 8

Due Friday, November 2 at the beginning of class (please submit your assignment on Canvas). Make sure to include your full name *and the list of your collaborators* (if any) with your assignment. You may discuss problems with others, but you may *not* keep a written record of your discussions. Please refer to the syllabus for details.

With regards to answering these problems, imagine that you are writing an answer to teach someone else in the class how to do the problem. In particular, you must give a complete outline for how you arrived at your answer. It is not sufficient to simply state a number or formula without providing the steps and reasoning that you used to produce the answer.

(1) Assume A is a set of sentences and ϕ, ψ are sentences. Prove (without using the completeness theorem) that the following are equivalent:
   - (a) $A \vdash \phi \wedge \psi$.
   - (b) $A \vdash \phi$ and $A \vdash \psi$.

(2) Assume ϕ is a sentence. Show (without using the completeness theorem) that $\vdash (\exists x)\neg\phi \leftrightarrow \neg(\forall x)\phi$. *Hint: do not hesitate to use the deduction theorem, the "proof by contradiction" lemma, as well as the introduction and elimination rules for quantifiers.*

(3) Let σ be a signature with at least one constant symbol. Assume A is a set of sentences in the language of σ. Solve the following two problems *without* using the completeness theorem.
   - (a) Let X be the set of all closed terms (i.e. terms without free variables) in the language of σ. Define a relation $\sim$ on X by $t \sim s$ if $A \vdash t = s$. Prove that $\sim$ is an equivalence relation.
   - (b) Let f be a function symbol of arity n and let $t_1, \ldots, t_n, s_1, \ldots, s_n$ be closed terms. Show that if $A \vdash t_i = s_i$ for all $i \leq n$, then $A \vdash f(t_1, \ldots, t_n) = f(s_1, \ldots, s_n)$.

(4) For a fixed signature σ and A, B sets of sentences in the language of σ, we write $A \vdash B$ if $A \vdash \phi$ for every $\phi \in B$.
   - (a) Prove without using the completeness theorem that if $A \vdash B$ and $B \vdash C$, then $A \vdash C$.
   - (b) Use the completeness theorem to give another proof that if $A \vdash B$ and $B \vdash C$, then $A \vdash C$.

Date: October 25, 2018.