MATH 141A: "NONSTANDARD" ANALYSIS

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Calculus was initially developped (especially by Leibniz) using the intuitive notion of an "infinitesimal" number: one that is "arbitrarily" close to zero, yet not zero. Expressions such as $\frac{\mathrm{d}y}{\mathrm{d}x}$ directly meant the quotient by an infinitesimal change of x of the corresponding infinitesimal change of y. That is, $\mathrm{d}y$ really does denote a true number, it is not just an idealization or a notation.

These infinitesimals (famously called "ghosts of departed quantities" by Berkeley) could never be made fully rigorous, and in the middle of the 18th century were replaced (by mathematicians such as Bolzano, Cauchy, and Weierstrass) by the ϵ-δ definitions that we all know and love! For example, a function $f : \mathbb{R} \to \mathbb{R}$ is defined to be *continuous* if for any real number a and any real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that for any real number x, $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$.

Using mathematical logic, we will see how Leibniz can get his revenge: we can make the notion of an infinitesimal rigorous and (many times) get rid of the ϵ and δ's, obtaining much simpler definitions and proofs in the process. This is often called "nonstandard" analysis, although many prefer terms such as *infinitesimal analysis* (what is standard analysis?). The presentation of these notes is inspired from H.J. Keisler's undergraduate calculus book (see links on the course website).

Let's fix a structure in which to do analysis: we are going to be greedy and work inside the structure R whose universe is $\mathbb{R}$, whose functions are *all* of the function $f : \mathbb{R}^n \to \mathbb{R}$, whose constant symbols are *all* of the real numbers, and whose relation symbols are *all* relations $S \subseteq \mathbb{R}^n$. In particular of course, we will have access to multiplication, addition, 0, 1, and the ordering on $<$. Now fix a nonprincipal ultrafilter U on ω. Define the *hyperreals* ${}^*R$ to be the ultrapower $\prod_{n\in\omega} R/U$ of R by U. Note that the definition really depends on U, so we should really talk about the *U-hyperreals*. However we suppress this from the notation because in practice the exact choice of U will never matter[1]. What are some of the properties of ${}^*R$?

First, you saw in problem 4 of assignment 7 that there is a canonical elementary embedding of R into ${}^*R$ (sending x to the class of the constantly-x function). Thus there is a copy of R inside ${}^*R$, so let's agree to identify R with the corresponding set of equivalence classes of constant functions in ${}^*R$. With this convention, R is an elementary substructure of ${}^*R$. In particular, the real numbers are a subset of the hyperreal numbers. We will write ${}^*\mathbb{R}$ for the set of hyperreal numbers, i.e. for the universe of ${}^*R$. Thus $\mathbb{R} \subseteq {}^*\mathbb{R}$.

[1] We could also look at ultrafilters on other sets than ω (for example on uncountable sets). Again, it usually does not make a difference what the index set is.

Further, given any function $s : \mathbb{R} \to \mathbb{R}$, there is a function symbol f_s that corresponds to it in the signature of R, and we can let ${}^*f$ be the interpretation of f_s in ${}^*R$. Explicitly, ${}^*f : {}^*\mathbb{R} \to {}^*\mathbb{R}$ will be defined by ${}^*f([(x_n)_{n<\omega}]) = [(f(x_n))_{n<\omega}]$.

In some sense, ${}^*f$ is a "natural" extension of the function f to the hyperreals. For example, the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ would be extended to the hyperreal function that just multiplies the number x with itself. Of course, we can do the same for functions of any finite number of variables. We will abuse notation and write e.g. $x + y$ for hyperreals x and y when we really mean $x {}^*{+} y$. We can also define objects such as ${}^*\mathbb{N}$ (take the interpretation of the unary predicate for $\mathbb{N}$ inside ${}^*R$). This ends up being $\{[(x_n)_{n<\omega}] \mid x_n \in \mathbb{N} \text{ for all } n < \omega\}$, a set of nonstandard natural numbers. It contains for example $a = [(n)_{n<\omega}]$ which by Łoś's theorem is above all the standard natural numbers. Thus for every $m < \omega$, ${}^*R \models m < a$. Now just like in R, division in ${}^*R$ is defined for every non-zero number. Thus it makes sense to let $\epsilon = \frac{1}{a}$, where division is done inside ${}^*R$. By the definition of a, for any natural number n, $\epsilon < \frac{1}{n}$. Since for any real number $x > 0$, there exists a natural number $n \in \mathbb{N}$ such that $x > \frac{1}{n}$, this means that ϵ is strictly less than any positive real number! We call such an ϵ an *infinitesimal*. More generally:

Definition 1. An hyperreal number x is:

(1) *Infinitesimal* if $|x| < r$ for any positive real number r.
(2) *Finite* if $|x| < n$ for some natural number n.
(3) *Infinite* if x is not finite.

Before proving some basic properties of the hyperreals, we restate Łoś's theorem:

Theorem 2 (The transfer principle)**.** For a sentence ϕ, $R \models \phi$ if and only if ${}^*R \models \phi$.

Note that the sentence ϕ may contain function symbols and constant symbols. When we interpret ϕ in ${}^*R$, the real-valued functions that ϕ mentions are "automatically extended" to their natural extensions in the hyperreals. For example $R \models (\forall x \forall y)|x + y| \leq |x| + |y|$. So ${}^*R \models (\forall x \forall y)|x + y| \leq |x| + |y|$, but in the second sentence, the quantifiers range over all *hyperreals*, and the absolute value function is the one extended to the hyperreals (i.e. if $f : \mathbb{R} \to \mathbb{R}$ is the function $f(x) = |x|$, then we are considering ${}^*f$ when talking about the triangle inequalities for hyperreals).

It is also important to note that ϕ has to be a first-order sentence, not just any statement whatsoever about the real numbers. For example, the *completeness axiom* of the real numbers, which says that any non-empty set of numbers which is bounded below has an infimum, is not true for the hyperreals (why?). This does not contradict the transfer principle because it is impossible to formalize the completeness axiom in first-order logic.

We can now prove some very basic properties of hyperreals:

Lemma 3. (1) For x a nonzero hyperreal, x is infinite if and only if $\frac{1}{x}$ is infinitesimal.
(2) If x and y are infinitesimals, then $x + y$ and $x \cdot y$ is infinitesimal.
(3) If $|x| \leq |y|$ and y is infinitesimal, then x is infinitesimal.

(4) If x and y are finite, then $x + y$ and $x \cdot y$ are finite.
(5) If x is finite and nonzero, and y is infinite, then xy is infinite.
(6) If x is finite and y is infinitesimal, then xy is infinitesimal.
(7) If x and y are positive and infinite, then $x + y$ and $x \cdot y$ are positive and infinite.

Proof. We leave most as exercises. Let us prove for example that if x and y are infinitesimals, then $x+y$ is infinitesimal. Fix a positive real number $r > 0$. Then by the triangle inequality (which transfers to the hyperreals by the transfer principle), $|x + y| \leq |x| + |y|$. By assumption, $|x| < \frac{r}{2}$ and $|y| < \frac{r}{2}$, so $|x| + |y| < r$. □

Every finite hyperreal number is infinitesimally close to a real number, in the following sense:

Definition 4. For hyperreal numbers x and y, we write $x \simeq y$ if $x - y$ is infinitesimal.

Lemma 5. The relation $\simeq$ is an equivalence relation on the hyperreals.

Proof. Clearly, $x - x = 0$ is an infinitesimal so $\simeq$ is reflexive. Also, a number α is infinitesimal if and only if $-\alpha$ is infinitesimal, so $\simeq$ is symmetric. Finally, if $x \simeq y \simeq z$, then by the triangle inequality, $|x - z| \leq |x - y| + |z - y|$. The right hand side is the sum of two infinitesimals, so $x - z$ must also be infinitesimal. □

We have arrived to another fundamental principle of nonstandard analysis:

Theorem 6 (The standard part principle)**.** For every finite hyperreal x, there exists a unique real number r such that $x \simeq r$.

Proof. First, if $x \simeq r_1$ and $x \simeq r_2$ for real numbers r_1 and r_2, then $r_1 \simeq r_2$, so $r_1 - r_2$ is infinitesimal. The only infinitesimal real number is zero, so $r_1 = r_2$. This shows uniqueness. To see existence, assume without loss of generality that $x > 0$ (if $x < 0$, replace x by $-x$, and note that $x \simeq r$ if and only if $-x \simeq -r$). Since x is finite, there is a minimal natural number n such that $x < n$. Let $X = \{r \in \mathbb{R} \mid x \leq r\}$. We know that X is not empty because $n \in X$. Also, X is bounded below (by $n-1$). By the completeness axiom for the real numbers, we can let $r = \inf(X)$. We claim that $x \simeq r$. Indeed, fix a real number $\epsilon > 0$. Then $r - \epsilon < x < r + \epsilon$ by definition of X and of the infimum, so $|x - r| < \epsilon$. Thus $x - r$ is infinitesimal. □

Definition 7. For a finite hyperreal number x, we let $\text{st}(x)$ (the *standard part of* x) denote the unique real number r such that $x \simeq r$.

We are now ready to start doing some calculus! Let's define continuity:

Definition 8. A function $f : \mathbb{R} \to \mathbb{R}$ is *continuous* if for any $a \in \mathbb{R}$ and any hyperreal x, $x \simeq a$ implies ${}^*f(a) \simeq {}^*f(x)$.

For example, $f(x) = x^2$ is continuous because if $x \simeq a$, then $x-a$ is infinitesimal, so $|x^2-a^2| = |(x-a)(x+a)|$ is an infinitesimal times a finite number, so is infinitesimal as well. Thus $x^2 \simeq a^2$. On the other hand, the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 0$ if $x < 0$ or $g(x) = 1$ if $x \geq 0$ is not continuous: $g(0) = 1$ but for a negative infinitesimal ϵ, $g(\epsilon) = 0 \not\simeq 1$.

Definition 9. A function $f : \mathbb{R} \to \mathbb{R}$ is *differentiable* at a real number a if for any non-zero infinitesimal ϵ the standard part of $\frac{f(a+\epsilon)-f(a)}{\epsilon}$ exists and does not depend on ϵ. We call this expression the derivative of f at a.

For example, the derivative of $f(x) = x^2$ at a point a is the standard part of:

$$\frac{(a+\epsilon)^2 - a^2}{\epsilon} = \frac{2a\epsilon + \epsilon^2}{\epsilon} = 2a + \epsilon$$

which is just $2a$. On the other hand the derivative of $f(x) = |x|$ at zero does not exist (because taking a positive or negative infinitesimal would change the standard part).

It may seem we are just doing the same arguments as the classical ones that use limits, but the simplifications can go further. For example, let's prove:

Theorem 10 (The extreme value theorem)**.** For any continuous function $f : \mathbb{R} \to \mathbb{R}$ and any real numbers $a \leq b$, f has a maximum on $[a, b]$.

Proof. For n a natural number and $\delta = \frac{b-a}{n}$, we always can split $[a, b]$ into $[a, a + \delta], [a + \delta, a + 2\delta], \ldots, [a + (n-1)\delta, a + n\delta]$. Since n is a natural number, we know that there exists $k \leq n$ such that $f(a + k\delta) \geq f(a + i\delta)$ for all $i \leq n$.

So now let N be an infinite natural number (i.e. an infinite member of ${}^*\mathbb{N}$) and let $\Delta = \frac{b-a}{N}$. We split ${}^*[a, b]$ into sets ${}^*[a, a + \Delta], {}^*[a + \Delta, a + 2\Delta], \ldots, {}^*[a + (N-1)\Delta, a + N\delta]$. By the transfer principle, there exists $K \leq N$ such that $f(a + K\Delta) \geq f(a + I\Delta)$ for every $I \leq N$. It is easy to see that ${}^*[a, b]$ is the set of hyperreals between a and b (inclusive), hence $a + K\Delta$ is finite, so let x be its standard part. We claim that x is a maximum of f in $[a, b]$. Indeed, take a real number $x' \in [a, b]$. Then $x' \in [a + I\Delta, a + (I + 1)\Delta]$ for some $I < N$. We have that these intervals have infinitesimal length, so since f is continuous, $f(x') \simeq f(a+I\Delta)$. On the other hand, $f(x) \simeq f(a + K\Delta)$, and $f(a + K\Delta) \geq f(a + I\Delta)$, so $f(x) \geq f(y)$, as desired. ☐

By the way, how is $\int_a^b f(x)dx$ defined? Well, if say f is continuous, take a positive infinitesimal dx, let $b' \simeq b$ be such that $b' - a$ is divisible by dx, and define $\int_a^b f(x)dx$ to be the standard part of:

$$\sum_{x=a,a+dx,a+2dx,\ldots,b'} f(x)dx$$

Thus in this definition $\int_a^b f(x)dx$ is really a sum of $f(x)dx$, where x moves across the interval $[a, b]$ by infinitesimal steps of dx. The fundamental theorem of calculus quite easily follows from this.

Much more can be done. For example one can do "Euler-style" analysis by working freely with sums of the form $\sum_{k=1}^{N} a_k$ where N is an infinite natural number[2].

[2]On this, see for example Mark McKinzie and Curtis Tuckey. *Higher Trigonometry, Hyperreal Numbers, and Euler's Analysis of Infinities.* Mathematics Magazine **74**, No. 5 (Dec., 2001), pp. 339–368.

Finally, when using this in an undergraduate calculus course there is of course no need to start with ultraproducts! One axiomatizes the hyperreals using the existence of infinitesimals, the transfer principle, the standard part principle and the extension principle (the fact that each function f extends to ${}^*f$) as given.