MATH 141A: THE INFINITE "PRISONNERS AND HATS" PUZZLE

A countable infinity of prisonners will be put inside a room. Each of the prisonner will have a hat, either black or white. Prisonners will be able to see the color of other people's hats but they won't see their own hat. The prisonners will be asked to all guess the color of their own hat at the same time. While in the room, they will not be allowed to communicate in any way. However they are allowed to discuss a strategy before entering the room.

We show that there is a way that all but finitely many prisonners can guess correctly.

For this, think of the prisonners as members of the set $\omega = \{0, 1, 2, \ldots\}$. An assignment of hat can be thought of as a function assigning a color to each member of ω. Thinking of black as 0 and white as 1, this means each assignment is a function $f : \omega \to 2$, i.e. $f \in {}^{\omega}2$. Consider the binary relation E on ${}^{\omega}2$ defined as follows: fEg if only if $\{n < \omega \mid f(n) \neq g(n)\}$ is finite. In other words, fEg if and only if f and g agree everywhere except for finitely-many places. Observe that E is an equivalence relation (exercise). Using the axiom of choice, pick a choice function $F : \mathcal{P}({}^{\omega}2) \backslash \{\emptyset\} \to {}^{\omega}2$. This function allows us in particular to pick a representative out of each E-equivalence class. The prisonners agree on F before entering the room.

Assume now the prisonners enter the room, and let f be the assignment of hats. Prisonner number n will know $f(m)$ for any $m \neq n$. Therefore it will know $[f]_E$ (but not f). Now let $g := F([f]_E)$, and have prisonner n guess the hat color $g(n)$. Since gEf, all but finitely-many prisonners will be correct.