MATH 141A: COMPACTNESS AND GRAPH COLORINGS

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We explore some connections between logic and combinatorics. First, an elementary principle: if we put n pigeons into k boxes and $k < n$, then one box will contain more than one pigeon (in fact at least $\lceil \frac{n}{k} \rceil$ pigeons). There are infinite versions of this that you will explore in the homework. Here, we consider Ramsey's theorem: a higher-dimensional generalization of the pigeonhole principle.

For a flavor of the statement, consider the following interesting fact: in a party with six students, there is always a group of three that either all know or all do not know each other (we assume knowledge is a symmetric irreflexive relation). Ramsey's theorem says that this is true more generally: for any size of group k, there exists a (very big) n such that in *any* party with n students there will be k students that either all know or all do not know each other (in the previous example, n was 6 and k was 3).

First, some notation:

Definition 1. For a set X and $m < \omega$, we write $[X]^m$ for the set of all subsets of X of cardinality exactly m.

Definition 2. Given a function $f : [X]^m \to Y$, we call a subset X_0 of X *homogeneous for* f if for any $a, b \in [X_0]^m$, $f(a) = f(b)$.

Typically, we think of f as a coloring of the pairs (or more generally m-tuples) of elements from X (where the coloring of a tuple does not depend on the order of its elements). A homogeneous set is one where all possible m-tuples from that set have the same color. In the example above, we were considering X to be the set of students in a party, $f(\{x, y\}) = 1$ if x and y know each other and $f(\{x, y\}) = 0$ otherwise. Then a homogeneous set is a set of students that either all know or all do not know each other.

Another way to think about this is that f colors the edges of a complete graph with set of vertices X. Then a homogeneous set is a set of vertices all of whose cross-edges have the same color.

One statement of the finite Ramsey theorem is[1]:

Theorem 3 (Finite Ramsey theorem)**.** For any $k < \omega$, there exists $n < \omega$ such that for any function $f : [n]^2 \to 2$, there is a homogeneous set $X \subseteq n$ of cardinality k.

[1] A more general statement allows coloring larger sets (i.e. "hyperedges") with more colors: for any $m < \omega$, any $c < \omega$, and any $k < \omega$, there exists $n < \omega$ such that for any function $f : [n]^m \to c$ there is a homogeneous set $X \subseteq n$ of cardinality k. For simplicity, we will focus on two colors and two dimensions.

Said less formally, for any $k < \omega$, there exists a big-enough $n < \omega$ such that any edge coloring of the complete graph on n vertices contains a complete graph on k vertices whose edges all have the same color.

Interestingly, the infinite version of Ramsey's theorem is both easier to state and easier to prove: in a party with infinitely-many students, there is an infinite group of students all of whose either know or do not know each other. Said more formally:

Theorem 4 (Infinite Ramsey theorem)**.** For any infinite set I and any $f : [I]^2 \to 2$, there exists an infinite homogeneous set for f.

Proof. While not really necessary, we give a proof using ultrafilters to show how they can make the combinatorics smoother. Fix U a nonprincipal ultrafilter on I. For each $x \in I$ and each $c < 2$, consider $N_{x,c} = \{y \in I - \{x\} \mid f(\{x, y\}) = c\}$, the set of neighbors of x whose corresponding edge is given color c by f. Note that $N_{x,0}, N_{x,1}, \{x\}$ form a partition of I, so exactly one set must be in U, and that set cannot be $\{x\}$, as U is not principal. Let $c_x < 2$ be the unique number such that $N_{x,c_x} \in U$. In words, c_x is the color of most outgoing edges of x. Now $\{x \in I \mid c_x = 0\}$, $\{x \in I \mid c_x = 1\}$ form a partition of ω, hence there is again a unique $c < 2$ such that $A = \{x \in I \mid c_x = c\} \in U$. In words, c is the color that most outgoing edges of most vertices have. We build a sequence $(x_n)_{n<\omega}$ of distinct elements of A by induction such that $f(\{x_n, x_m\}) = c$ for all $m < n$. This will be enough, as then $\{x_n : n < \omega\}$ is homogeneous for f.

Since U is a filter, A is not empty, so pick $x_0 \in A$. Now given x_n, let $A_n = A \cap \bigcap_{m \le n} N_{x_m,c}$. Note that $A_n \in U$, since U is closed under finite intersections. Since U is not principal, we can pick $x_{n+1} \in A_n - \{x_m \mid m \le n\}$. By definition of A_n, $f(\{x_m, x_{n+1}\}) = c$ for all $m < n + 1$, as desired. □

As a quick application, we obtain a short proof of the Bolzano-Weierstrass theorem:

Corollary 5. Any bounded sequence of real numbers has a convergent subsequence.

Proof. Let $(a_n)_{n<\omega}$ be a bounded sequence of real numbers. We define a function $f : [\omega]^2 \to 2$ by (when $n < m$) $f(\{n, m\}) = 1$ if $a_n < a_m$ and $f(\{n, m\}) = 0$ otherwise. By the infinite Ramsey theorem, there is an infinite homogeneous set X for f. If the color on unordered pairs from X is 1, then for $n < m$ in X, $a_n < a_m$. If the color is 0 then for $n < m$ in X, $a_n \ge a_m$. Either way, $(a_n)_{n \in X}$ is a monotone sequence and it is bounded by assumption, hence converges. □

Now we show that the infinite Ramsey theorem together with the compactness theorem directly implies the finite Ramsey theorem. In fact, a strengthened version of Ramsey's theorem (which is harder to prove directly) also follows.

Theorem 6 (Strengthened finite Ramsey theorem)**.** For any $k < \omega$ there exists $n < \omega$ such that for any function $f : [n]^2 \to 2$, there is a homogeneous set $X \subseteq n$ of cardinality at least k with $\min(X) \le |X|$.

Proof. Suppose not. Then there exists $k < \omega$ such that for all $n < \omega$, there exists $f : [n]^2 \to 2$ such that any set $X \subseteq n$ of cardinality at least k with $\min(X) \le |X|$ is not homogeneous for f. Work in the signature $\sigma = \{c_0, c_1, f\} \cup \{d_i : i < \omega\}$, where

c_0 and c_1 are constant symbols, each d_i is a constant symbol, and f is a binary function symbol. Consider the set A of axioms with the following sentences:

(1) $c_0 \neq c_1$.
(2) $d_i \neq d_j$, (for all $i \neq j$ in ω).
(3) $(\forall x)(f(x) = c_0 \vee f(x) = c_1)$.
(4) $(\forall x)(f(x, x) = c_0)$ (the value of f at a repeated coordinate is irrelevant).
(5) $(\forall x \forall y)(f(x, y) = f(y, x))$.
(6) For each fixed finite $X \subseteq \omega$, with $|X| \geq k$ and $\min(X) \leq |X|$, add the sentence:

$$\bigvee_{i,i',j,j' \in X, i \neq j, i' \neq j'} f(d_i, d_j) \neq f(d_{i'}, d_{j'})$$

(that is, the set $\{d_m \mid m \in X\}$ is not homogeneous for f).

By our assumption on k, every finite subset of A is consistent. Therefore by the compactness theorem also A is consistent. Let $M \models A$. Let $D := \{d_i^M \mid i < \omega\}$. By the infinite Ramsey theorem, there is an infinite homogeneous set $X \subseteq D$ for $f^M \upharpoonright D$. Let $m < \omega$ be minimal such that $d_m^M \in X$. Since X is infinite, we can pick a finite $X_0 \subseteq X$ with $|X_0| \geq m + k$. Then for any $i \neq j$, $i' \neq j'$ all in X_0, $f^M(d_i^M, d_j^M) = f^M(d_{i'}^M, d_{j'}^M)$, a contradiction to the last part of the definition of A. □

Remark 7. By a result known as the *Paris-Harrington theorem*, the strong finite Ramsey theorem is *not* a consequence of PA (to formalize it, one would need to first code finite sets of natural numbers as single natural numbers). Intuitively, this is because the function sending $k < \omega$ to the least $n < \omega$ satisfying the conclusion of the strong Ramsey theorem grows so fast that PA cannot prove it is bounded (on the other hand, the regular finite Ramsey theorem is a consequence of PA – for a fixed k, one can bound the least $n < \omega$ satisfying its conclusion by an "elementary" function, namely 2^{2^k}). Thus the strong finite Ramsey theorem is an example of a natural statement witnessing that PA is incomplete!

Instead of coloring edges of a complete graph, we may want to color vertices of a (possibly incomplete) graph and ask that no edge links vertices with the same color:

Definition 8. If $G = (V, E)$ is a graph and k is a natural number, a *k-coloring* of G is a function $f : V \to k$ such that for any $(x, y) \in E$, $f(x) \neq f(y)$. We say that G is *k-colorable* if there is a k-coloring of G. The *coloring number* of G is the minimal k such that G is k-colorable.

The study of coloring number of finite graphs is an active part of combinatorics. What about infinite graphs? Using the compactness theorem again, one can show that it is enough to consider all the finite subgraphs (a *subgraph* of a graph (V, E) is a graph (V_0, E_0) such that $V_0 \subseteq V$ and $E_0 \subseteq E$)! You will prove this in your homework.

Exercise 9 (De Bruijn–Erdős theorem)**.** For a natural number k, a graph is k-colorable if and only if all of its finite subgraphs are k-colorable.

As a sample application, consider the *Hadwiger-Nelson problem*: define a graph $G = (V, E)$, where $V = \mathbb{R}^2$ and xEy if only if the Euclidean distance between x and y is exactly one. The *coloring number of the plane* is defined to be the coloring number of G.

What is the coloring number of the plane? While G is continuum-sized, and hence may seem very complicated, by the De Bruijn-Erdős theorem, it is enough to figure out the coloring number of any *finite* subgraph of G: any finite arrangement of points in the plane, where there is an edge between two points if they are at distance exactly one.

Until recently, all that was known was that the coloring number of the plane is at most 7 (the plane can be covered by regular hexagons of diameters slightly less than one, colored in a certain way), and at least 4 (there is a 10 vertices example called the *Moser spindle* which cannot be 3-colored). Recently (last semester), Aubrey de Grey, a biologist at MIT, used a clever construction (and the help of a computer!) to find a subgraph of G with 1581 vertices which cannot be 4-colored. Thus the coloring number of the plane is at least 5. It remains open whether the coloring number of the plane is 5, 6, or 7... The wikipedia article[2] on this problem has more information, including some nice pictures.

[2] `https://en.wikipedia.org/wiki/Hadwiger-Nelson_problem`