# MATH 141A: SKOLEM'S PARADOX

Recall Löwenheim's theorem (Theorem 2.5 in Poizat): any relation has a countable elementary restriction. Let's apply this to the membership relation in sets.

More precisely, let $V$ be the class of all sets, and consider the binary relation $R$ on $V$ defined by $xRy$ if and only if $x \in y$. By Löwenheim's theorem (it does not matter that $V$ is not a set: the proof goes through), $(V, R)$ has a countable elementary restriction $(V_0, (R \cap V_0 \times V_0))$. You can think of $V_0$ as a small model of set theory: it satisfies every single property you can think of about sets, but (as opposed to $V$ which is not even a set) it is countable! How is that possible? Isn't "there exists uncountable sets" a property of sets? Let's backtrack a bit.

First note for example that many sets you know are in $V_0$: for example, the empty set is the unique set $x$ satisfying the formula $\forall y \neg r(y, x)$. Therefore it must be in $V_0$, as $V_0$ is an elementary restriction of $V$. Similarly, any set that can be defined explicitly using a formula will be in $V_0$. For example, $\omega$ is the unique set $x$ so that $\emptyset \in x$, $y \in x$ implies $y \cup \{y\} \in x$, and for any other set $x'$ satisfying these properties, $x \subseteq x'$. This can be readily translated into a formula. Similarly, $\mathbb{R}$, the set of real numbers, can be defined by a (pretty long) formula: starting with $\omega$, construct the integers (for example as pairs $(\omega, \ell)$ with $\ell = 1, 2$ – note that pairs can be coded as sets: $(x, y)$ can be defined to be $\{\{x\}, \{x, y\}\}$), then the rationals e.g. as equivalence classes of pairs of integers, and finally the reals e.g. as Dedekind cuts of rationals or equivalence classes of Cauchy sequences of rationals.

We have just argued that $\mathbb{R} \in V_0$ and we know[1] that $\mathbb{R}$ is uncountable. Note that $\mathbb{R} \cap V_0$ must be countable, as $V_0$ is countable. Thus $V_0$ has only countably-many reals. This may seem a little bit strange but now comes the real "paradox": There is a (again very long) sentence that expresses "$\mathbb{R}$ is uncountable": just say that there is no surjection from $\omega$ onto $\mathbb{R}$. Since $V \models$ "$\mathbb{R}$ is uncountable$''$ and $V_0$ is an elementary restriction, it must satisfy this too: $V_0 \models$ "$\mathbb{R}$ is uncountable$''$. How can $V_0$ think there are uncountably-many reals if it has only countably-many?

The resolution is that $V_0$ is "wrong" about countability: while indeed there is a surjection $s$ from $\omega$ onto $\mathbb{R} \cap V_0$, and $s \in V$, we will *not* have that $s \in V_0$. This does not contradict elementarity, as $\mathbb{R} \cap V_0$ is not a member of $V_0$ and it turns out that it is impossible to define $s$ using only parameters from $V_0$.

[1] One could replace $\mathbb{R}$ by $\mathcal{P}(\omega)$ or any other definable uncountable set.