MATH 145A - SET THEORY I, FALL 2019
ASSIGNMENT 4

Due Tuesday, October 1 at the beginning of class (please submit your assignment as a PDF on Canvas). Make sure to include your full name *and the list of your collaborators* (if any) with your assignment. You may discuss problems with others, but you may *not* keep a written record of your discussions. Please refer to the syllabus for details.

As a general rule, imagine that you are writing your solution to convince somebody else in the class who is very skeptical about the particular statement. In particular, it should be completely understandable to another student: always justify your reasoning in plain English. It is not sufficient to simply state a number or formula without providing the steps and reasoning that you used to produce the answer.

(1) (a) Show that $\alpha + \beta$ is isomorphic to the concatenation $(\alpha, \in) \oplus (\beta, \in)$.
  (b) Show that $\alpha\beta$ is isomorphic to $(\beta, \in) \times (\alpha, \in)$, ordered with the lexicographic ordering.

(2) Prove the following basic properties of ordinal arithmetic, and give examples showing that the left versions of these properties are not true in general.
  (a) Right (strict) monotonicity: if $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$.
  (b) Right continuity: if X is a non-empty set of ordinals, $\sup_{\beta \in X}(\alpha+\beta) = \alpha + \sup_{\beta \in X} \beta$.
  (c) Right cancellation: if $\alpha + \beta = \alpha + \gamma$, then $\beta = \gamma$.
  (d) Right distributivity: $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.

(3) (a) Show that any ordinal α can be written as $\alpha = \delta + n$ for $n < \omega$ and δ limit or zero.
  (b) Show that for any ordinal α and any ordinal $\beta > 0$, there exists unique ordinals γ and δ such that $\delta < \beta$ and $\alpha = \beta\gamma + \delta$ *Note: this shows there is a kind of division. Hint: take γ minimal such that $\beta(\gamma + 1) > \alpha$, then find δ.*
  (c) Let $\gamma \geq 2$ be an ordinal. Show that for any ordinal α, there exists $n < \omega$ and ordinals $\alpha_0 > \alpha_1 > \ldots > \alpha_{n-1}$, $c_0, \ldots, c_{n-1}$ such that $0 < c_i < \gamma$ for all $i < n$ and $\alpha = \gamma^{\alpha_0} c_0 + \gamma^{\alpha_1} c_1 + \ldots + \gamma^{\alpha_{n-1}} c_{n-1}$ (with the convention that when $n = 0$, the empty sum is zero). *Note: This representation is called the base γ normal form of α. It is tedious but possible to show that this representation is also unique. Hint: proceed by induction on α: to start, take α_0 to be minimal such that $\gamma^{\alpha_0+1} > \alpha$ and "divide" by γ^{α_0}.*
  (d) Write the following expression in base ω normal form (no full proof needed, you only need to justify briefly). For example, $\omega + \omega = \omega^1 \cdot 2$ in base ω normal form.
    (i) $3 + \omega + 5 + \omega$.
    (ii) $\omega^2 \cdot 3 + 5 \cdot \omega^3$.
    (iii) $(\omega \cdot 3) \cdot (\omega \cdot 5)$.
    (iv) $2^{32 \cdot (1+\omega) + 2^\omega}$ (*Warning: exponentiation here is ordinal, not cardinal, exponentiation.*)

(4) Show that for sets X and Y, the following are equivalent:
  - $|X| \leq |Y|$.
  - There is an injection from X to Y.
  - $X = \emptyset$ or there is a surjection from Y to X.

(5) Show (without using the axiom of choice) that the following are equivalent:
  - The axiom of choice.
  - There is a bijection from OR to SET.

Date: September 23, 2019.

Deduce that a class A is proper if and only if there is a bijection from A to SET. *Hint: think about the cumulative hierarchy.*

(6) (Extra credit) Zorn's lemma is the statement that any ordering in which every chain has an upper bound has a maximal element (see the notes for definitions of these terms).
  (a) Show (without using the axiom of choice) that the following are equivalent.
    - The axiom of local choice.
    - For every set X, there is a well-ordering on X.
    - Zorn's lemma.
  (b) *Zorn's lemma for classes* states that if P is a class ordering with no maximal elements in which every (set) chain has an upper bound, then P contains a chain which is a proper class.
    (i) Explain why Zorn's lemma for classes implies Zorn's lemma.
    (ii) Prove that Zorn's lemma for classes is equivalent to the axiom of choice.