MATH 145A - SET THEORY I, FALL 2019
ASSIGNMENT 5

Due Tuesday, October 8 at the beginning of class (please submit your assignment as a PDF on Canvas). Make sure to include your full name *and the list of your collaborators* (if any) with your assignment. You may discuss problems with others, but you may *not* keep a written record of your discussions. Please refer to the syllabus for details.

As a general rule, imagine that you are writing your solution to convince somebody else in the class who is very skeptical about the particular statement. In particular, it should be completely understandable to another student: always justify your reasoning in plain English. It is not sufficient to simply state a number or formula without providing the steps and reasoning that you used to produce the answer.

(1) Let $(\lambda_i)_{i \in I}$ be a sequence of infinite cardinals. Show that:

$$\sum_{i \in I} \lambda_i = \max(|I|, \sup_{i \in I} \lambda_i)$$

(2) (a) Let $I = (A, \leq)$ be any linear order and let θ be the cofinality of I.
  - (i) Prove that there exists a sequence $(x_i)_{i<\theta}$ of elements of A such that $i < j < \theta$ implies $x_i < x_j$, and $\{x_i \mid i < \theta\}$ is cofinal in I. Such a sequence is called a *cofinal sequence* in I.
  - (ii) Deduce that θ is a regular cardinal.

  (b) Let δ be a limit ordinal. Show that $\mathrm{cf}(\aleph_\delta) = \mathrm{cf}(\delta)$.

  (c) Let λ be a limit cardinal of cofinality θ. Prove that there exists a cofinal sequence $(\lambda_i)_{i<\theta}$, where each λ_i is a *cardinal* for each $i < \theta$.

(3) Give the cardinality of each of the sets below (and justify). Your answer can only involve cardinals of the form $\aleph_\alpha$, for α an ordinal, and the function $\lambda \mapsto 2^\lambda$. For example, $|\mathbb{R}| = 2^{\aleph_0}$ is an acceptable answer, but $|\mathbb{R}| = \aleph_0^{\aleph_0}$ is not.
  - (a) The cube $[0, 1] \times [0, 1] \times [0, 1]$.
  - (b) The Hilbert cube $\prod_{n \in \mathbb{N} - \{0\}} [0, \frac{1}{n}]$.
  - (c) The long line $\omega_1 \times \mathbb{R}$.
  - (d) The very high-dimensional cube ${}^{\omega_1 \times \mathbb{R}}[0, 1]$.
  - (e) The set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$.
  - (f) The set of all functions from $\mathbb{R}$ to $\mathbb{R}$.
  - (g) The set of all subsets of $\mathbb{R}$.
  - (h) The set of all Borel subsets of $\mathbb{R}$. *Hint: first count the number of open subsets of* $\mathbb{R}$.

(4) Prove *Hausdorff's formula*: for any infinite cardinals λ and μ:

$$\left(\lambda^+\right)^\mu = \lambda^+ \cdot \lambda^\mu$$

(5) Let $F : \mathrm{OR} \to \mathrm{OR}$ be a class function. We say that F is *continuous* if for any limit ordinal δ, $F(\delta) = \sup_{\alpha<\delta} F(\alpha)$. F is *strictly increasing* if $\alpha < \beta$ implies $F(\alpha) < F(\beta)$. A *fixed point* for F is an ordinal α such that $F(\alpha) = \alpha$.
  - (a) Prove that if F is strictly increasing, then $\alpha \leq F(\alpha)$ for any ordinal α.
  - (b) Prove that if F is continuous and strictly increasing, then it has a fixed point. *Hint: keep applying F to itself until you get to the fixed point.*
  - (c) Deduce that there exists a cardinal λ such that $\lambda = \aleph_\lambda$.
  - (d) What is the cofinality of the λ you found in the previous part?

Date: October 1, 2019.

(6) (Extra credit) Prove that for all $\alpha < \omega_1$, there is an order embedding of α into $\mathbb{R}$ (with the usual orderings). Prove on the other hand that there is no order embedding of ω_1 into $\mathbb{R}$.