MATH 145A - SET THEORY I, FALL 2019
ASSIGNMENT 9

Due Wednesday, Nov. 6, 2019, 11h59pm. (please submit your assignment as a PDF on Canvas). Make sure to include your full name *and the list of your collaborators* (if any) with your assignment. You may discuss problems with others, but you may *not* keep a written record of your discussions. Please refer to the syllabus for details.

As a general rule, imagine that you are writing your solution to convince somebody else in the class who is very skeptical about the particular statement. In particular, it should be completely understandable to another student: always justify your reasoning in plain English. It is not sufficient to simply state a number or formula without providing the steps and reasoning that you used to produce the answer.

(1) Recall that a filter F on a set S is *principal* if there exists a subset A of S such that $F = \{X \subseteq S \mid A \subseteq X\}$. Prove that any filter on a finite set is principal.

(2) A *near-partition* of a set S is a collection P of pairwise disjoint subsets of S such that the union of all sets in P is S (the difference with a partition is that we allow the empty set to be in P). Prove that the following are equivalent, for a non-empty set S and a collection U of subsets of S:
   (a) U is an ultrafilter on S.
   (b) $|U \cap P| = 1$ for any near-partition P of S with $|P| \leq 3$.
   (c) $S \in U$, U is closed under taking supersets ($A \in U$, $A \subseteq B \subseteq S$ implies $B \in U$), and for any two disjoint subsets A and B of S, if $A \cup B \in U$ then exactly one of A or B is in U.

(3) Let F be a filter on ω. Let $(a_n)_{n<\omega}$ be a sequence of real numbers and let a be a real number. Recall that $(a_n)_{n<\omega}$ *F-converges to a* if for any $\epsilon > 0$, $\{n < \omega \mid |a_n - a| < \epsilon\} \in F$. We call a an *F-limit* of $(a_n)_{n<\omega}$ and write $\lim_F a_n = a$.
   (a) Show that if $(a_n)_{n<\omega}$ F-converges to both a and b, then $a = b$. *Note: this shows that the F-limit is unique if it exists.*
   (b) Describe the F-limit if F is the cofinite filter. What if F is a principal filter?
   (c) Assume now that U is an ultrafilter. Show that any bounded sequence U-converges (a sequence (a_n) is *bounded* if there exists $C > 0$ so that $|a_n| < C$ for all n).
   (d) If $\lim_F a_n = a$ and $\lim_F b_n = b$, then:
      (i) $\lim_F (a_n + b_n) = a + b$.
      (ii) For any $c \in \mathbb{R}$, $\lim_F ca_n = ca$.
      (iii) If $a_n \leq b_n$ for all n, then $a \leq b$.

(4) Let λ be a regular uncountable cardinal. Assume $(C_i)_{i<\lambda}$ are club subsets of λ.
   (a) Show that for any nonzero $\alpha < \lambda$, $\bigcap_{i<\alpha} C_i$ is club.
   (b) Show that $\triangle_{i<\lambda} C_i$ is club.

(5) *(Extra credit)* Is there a way to associate to each bounded sequence $(a_n)_{n\in\mathbb{N}}$ a "limit" $\lim^*_{n\to\infty} a_n$ such that the following properties are satisfied?
   - $\lim^*_{n\to\infty} a_n = a$ if $\lim_{n\to\infty} a_n = a$.
   - $\lim^*_{n\to\infty} (a_n + b_n) = \lim^*_{n\to\infty} a_n + \lim^*_{n\to\infty} b_n$.
   - For any $c \in \mathbb{R}$, $\lim^*_{n\to\infty} ca_n = c \lim^*_{n\to\infty} a_n$.
   - (Translation invariance) For any $m \in \mathbb{N}$, $\lim^*_{n\to\infty} a_n = \lim^*_{n\to\infty} a_{n+m}$.

Date: October 30, 2019.