THE ERDŐS-KO-RADO THEOREM

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We present one more application of the probabilistic method.

Definition. Let X be a set. A family $\mathcal{F}$ of subsets of X is *intersecting* if $A \cap B \neq \emptyset$ for any $A, B \in \mathcal{F}$.

If X has n elements, how big can an intersecting family be? Suppose in particular that we require the members of $\mathcal{F}$ to all have size k. One simple construction is to fix $a \in X$ and let $\mathcal{F}$ be the set of all k-elements subsets of X that contain a. Then $\mathcal{F}$ is clearly intersecting, and has size $\binom{n-1}{k-1}$. The next result shows that we cannot do better:

Theorem (Erdős-Ko-Rado)**.** Let X be an n-element set and let k be a natural number such that $2k \leq n$. If $\mathcal{F}$ is an intersecting family of k-element subsets of X, then:

$$|\mathcal{F}| \leq \binom{n-1}{k-1}$$

Proof. Clearly, it does not matter what the elements of X actually are, so assume without loss of generality that $X = \{0, 1, \ldots, n-1\}$. Let $\mathcal{F}$ be an intersecting family of k-element subsets of X. For a fixed $i \in X$, let $A_i := \{i, i+1, \ldots, i+k-1\}$ (where addition is done modulo n). This is about the simplest type of subset of X that one can think of. How many subsets of this form can there be in an intersecting family? At most k: since $2k \leq n$, $A_i \cap A_{i+k} = \emptyset$ for each $i \in X$. So assume that $A_i \in \mathcal{F}$, and let $k_1 < k$ be maximal such that A_{i-k_1} is in $\mathcal{F}$, and similarly let $k_2 < k$ be maximal such that A_{i+k_2} is in $\mathcal{F}$. We then have that $k_2 - k_1 < k$, and $A_j \in \mathcal{F}$ implies that $j \in \{i - k_1, i - k_1 + 1, \ldots, i + k_2\}$.

Next, observe that *any* k-element subset of X can be obtained by applying a permutation of X to some A_i. Thus suppose we take a random permutation π of X and also independently choose a random $i \in X$. What is the probability p that the set $\pi(A_i) := \{\pi(a) \mid a \in A_i\}$ is in $\mathcal{F}$? We can estimate it in two different ways. On the one hand, we have just seen that for a fixed π, there are at most k out of n possible values of i so that $\pi(A_i) \in \mathcal{F}$. Thus $p \leq \frac{k}{n}$. On the other hand it is also possible to compute p directly: it is just the number of sets in $\mathcal{F}$ divided by the total number of k-element subsets: $p = \frac{|\mathcal{F}|}{\binom{n}{k}}$.

Putting these two estimates together, we get that $\frac{|\mathcal{F}|}{\binom{n}{k}} \leq \frac{k}{n}$, so $|\mathcal{F}| \leq \frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}$, as desired. $\square$