MATH 155R - COMBINATORICS, FALL 2019
ASSIGNMENT 6

Due Tuesday, October 15, before 11h59pm (please submit your assignment as a PDF on Canvas). Make sure to include your full name *and the list of your collaborators* (if any) with your assignment. You may discuss problems with others, but you may *not* keep a written record of your discussions. You may also freely look at the hints at the end of MN[1]. Please refer to the syllabus for details.

As a general rule, imagine that you are writing your solution to convince somebody else in the class who is very skeptical about the particular statement. In particular, it should be completely understandable to another student: always justify your reasoning in plain English. It is not sufficient to simply state a number or formula without providing the steps and reasoning that you used to produce the answer.

(1) (MN, 5.1.2) Prove that any graph $G = (V, E)$ having no cycles and satisfying $|V| = |E| + 1$ is a tree.

(2) Let $k \geq 2$, let T be a tree on k vertices and let G be a graph with minimum degree at least $k-1$. Prove that T is isomorphic to a subgraph of G.

(3) Let G be a graph. An *independent set* in G is a set I of vertices that are not connected by an edge: if $x, y \in I$ then $\{x, y\} \notin E(G)$. A *vertex cover* of G is a set C of vertices so that for every edge $e \in E(G)$ there exists $v \in C$ with $v \in e$.

Let $\alpha(G)$ denote the maximal size of an independent set in G, and let $\tau(G)$ denote the minimal size of a vertex cover of G.

(a) Give a formula for $\tau(G)$ in terms of $\alpha(G)$.

(b) Let G be a triangle-free graph. Show that the degree of any vertex in G is at most $\alpha(G)$.

(c) Prove that for every triangle-free graph G, $|E(G)| \leq \alpha(G)\tau(G)$.

(d) Explain why this implies that for every triangle-free graph G, $|E(G)| \leq \frac{|V(G)|^2}{4}$.

(4) (MN, 6.2.(b)) Draw K_6, the complete graph on six vertices, on the torus in such a way that no edges cross.

Date: October 7, 2019.

[1]Matoušek and Nešetřil, *Invitation to discrete mathematics*, 2nd edition, Oxford University Press, 2008.