CONCEPTS OF MATHEMATICS, SUMMER 1 2014
ASSIGNMENT 1

Due Friday, May 23 at the beginning of class. Make sure to include your name, Andrew ID, *and the list of your collaborators* (if any) with your assignment. You may discuss problems with others, but you may *not* keep a written record of your discussions. Please refer to the syllabus for details.

Extra credit 1 (10 points): Tell me about yourself!

There are of course no right or wrong answers for this part: anybody filling it in will get full credit! *Please hand it in separately from the rest of your assignment.*

It is useful for me to know about the background of my students, and it may also be useful for you to clarify your expectations for this course.

(1) First, fill-in the following grid so that I can schedule my office hours at a good time. *Mark times at which you would be able to come to office hours if needed.* Do *not* mark times at which e.g. you have another lecture.

	Monday	Tuesday	Wednesday	Thursday	Friday
8h30-9h30					
9h30-10h30					
10h30-11h30					
11h30-12h30					
12h30-13h30					
13h30-14h30					
14h30-15h30					
15h30-16h30					
16h30-17h30					
17h30-18h30					
18h30-19h30					
19h30-20h30					

(2) What does "mathematics" mean to you?

Date: May 7, 2014.

(3) Tell me about your "math education". For example, what courses in mathematics or science have you taken before?
(4) Why are you taking 21-127? What do you expect to get out of it?

Problem 1 (20 points)

Recall that π is defined to be the ratio of a circle's circumference to its diameter (you may take it for granted that this ratio is always constant). A well known theorem of Archimedes says that the *area* of a circle of radius r is given by πr^2 (this is actually not that easy to prove). Show that $\pi \neq 4$.

You are allowed to use basic geometric facts and constructions, but not any results (except the ones just mentioned) about π itself. If you draw a picture, explain precisely how it is drawn. If you want to claim a quantity is greater than another, make sure you justify why without relying too much on a (possibly imperfect) drawing. You are not expected to achieve the same level of preciseness as for the other problems, but try to justify each step as well as you can.

Problem 2 (40 points)

(1) What is wrong with the following "proof" that $0 = 1$?
Assume x, y are real numbers such that $x = y$.

$$\begin{aligned} x &= y && \text{(by assumption)} \\ x - y &= 0 && \text{(Subtracting } y \text{ from both sides)} \\ (x-y)(x-y)^{-1} &= 0(x-y)^{-1} && \text{(Multiplying both sides by } (x-y)^{-1}) \\ 1 &= 0(x-y)^{-1} && (1 = (x-y)(x-y)^{-1} \text{ by definition of the reciprocal)} \\ 1 &= 0 && \text{(Any number multiplied by zero is zero } (F_0)) \end{aligned}$$

(2) What is wrong with the following "proof" that all nonzero real numbers are positive?
For x a nonzero real number, $0 < x \cdot x$ (by F_{15}). Multiply both sides of the inequality by x^{-1} to obtain $0 \cdot x^{-1} < x \cdot x \cdot x^{-1}$. Since any number multiplied by zero is zero (F_0), $0 < x \cdot x \cdot x^{-1}$. By definition of the reciprocal, $x \cdot x^{-1} = 1$, so $0 < x \cdot 1$. Since one is the multiplicative identity (M_2), $0 < x$.
(3) Notice that the wrong proof of the first part does not by itself *disprove* that $0 = 1$ (one can always give wrong proofs of true facts). Using only the axioms and facts about real numbers

from the lecture notes, prove that $0 \neq 1$. *Hint:* This should be very easy.

(4) Show that not all nonzero real numbers are positive by giving an explicit example of a negative real number. Of course, you should prove (using only the axioms and facts of real numbers from the lecture notes, as usual) that your example is indeed negative.

Problem 3 (20 points)

Assume x, y are real numbers with $x < y$. Find a real number z such that $x < z < y$ (of course, you should *prove* that your z has this property).

Problem 4 (20 points)

(1) Using only the axioms of real numbers, prove (F_0): For any real number x, $x \cdot 0 = 0$. *Hint:* Write 0 as $0 + 0$.
(2) Using only the axioms of real numbers, prove (F_4): For all real numbers x, y, if $xy = 0$, then $x = 0$ or $y = 0$.
(3) Using only the axioms of real numbers, prove that $(-1)(-1) = 1$. *Hint:* Show that $(-1)(-1) + (-1) = 0$.
(4) Using only the axioms of real numbers, prove (F_{10}): $0 < 1$.

Extra credit 2 (20 points): A paradox

What is wrong with the following argument?

By "characters", we mean letters from the roman alphabet: a, ..., z, A, ..., Z, digits: 0, 1, ..., 9, spaces, dots, quotation marks: ",", brackets: (,), or commas. Some real numbers can be defined using relatively few characters. For example, a definition of "pi" would be "the ratio of a circle's circumference to its diameter". However, there are only finitely many characters, so one can form only finitely many sentences of at most 42000 characters. Therefore only finitely many real numbers can be defined using sentences of at most 42000 characters. It follows that there must be a *largest* real number x definable in this way. Let y be this number plus one.

The previous paragraph gives a definition of y of less than 42000 characters, yet y cannot be bigger than x by definition of x, a contradiction.