CONCEPTS OF MATHEMATICS, SUMMER 1 2014
ASSIGNMENT 10

Due Thursday, June 26 at the beginning of class. Make sure to include your name, Andrew ID, *and the list of your collaborators* (if any) with your assignment. You may discuss problems with others, but you may *not* keep a written record of your discussions. Please refer to the syllabus for details.

Instructions: Do Problem 1 and any two of the remaining four non-extra-credit problems. Clearly mark which two problems you chose. Of course, you are encouraged to also solve the remaining problems (for glory). The sixth problem is a hard extra credit worth 40 points!

Problem 1 (20 points)

A secretive three letter government agency decides to question all its employees with a lie detector. We assume that there is a 0.001 probability that the lie detector thinks you are lying when you are really telling the truth (this is called a *false positive*). Otherwise, we assume that the lie detector always detects when somebody is lying. Assume the agency has 100 000 employees and knows for sure that exactly 10 of them are dishonest.

An employee takes the test and makes the lie detector beep. What is the probability that the employee is really lying?

Problem 2 (40 points)

Assume we are working in a finite probability space S with probability function P. Assume A and B are events. Prove or disprove:

(1) If $A \subseteq B$, then $P(A) \leq P(B)$.
(2) If $P(B) \neq 0$ and $P(A|B) = P(A)$, then A and B are independent.
(3) If A and B are independent, then A^c and B^c are independent.
(4) If $P(A) > 1/2$ and $P(B) > 1/2$, then $P(A \cap B) > 0$.

Problem 3 (40 points)

We toss n coins one after the other (where n is a positive natural number). Assume that for each coin, the probability that it falls on head is p, $0 \leq p \leq 1$.

(1) Give a finite probability space modeling this experiment. Don't forget to define the probability function P.
(2) Assume k is a natural number. What is the probability of obtaining exactly k heads?

Problem 4 (40 points)

Assume p_0, p_1, ..., p_n are distinct primes. Let S be the set of all natural numbers of the form $p_0^{k_0} p_1^{k_1} \dots p_n^{k_n}$, where each k_i is a natural number, $0 \leq k_i \leq 42$. For $k \leq 42$ a natural number, what is the probability that a randomly chosen element of S is a product of exactly k (not necessarily distinct) primes?

Date: June 24, 2014.

Problem 5 (40 points)

(1) Assume S is a finite probability space with probability function P. Assume A_0, A_1, ..., A_n are events and n is a natural number. Prove the *union bound:* $P(A_0 \cup A_1 \cup \ldots \cup A_n) \leq \sum_{i=0}^{n} P(A_i)$.

(2) Assume m and n are natural numbers and S is the set ${}^{[n]}[m]$ of functions from $[n]$ to $[m]$. Assume that $n \leq \sqrt{m}$. Show that if we choose a function $f \in S$ uniformly at random, the probability that f is injective is at least $1/2$. *Hint: write the event that a randomly selected f is not injective as $\bigcup_{i<j} A_{i,j}$ where $A_{i,j}$ is the event that $f(i) = f(j)$.*

Extra credit (40 points): Infinite probability spaces

In this exercise, you will show why infinite probability spaces are tricky to define right!

Assume S is an infinite set and $P : \mathcal{P}(S) \to \mathbb{R}$ is a function satisfying the following properties:

- If $A \subseteq S$, $0 \leq P(A) \leq 1$.
- $P(S) = 1$.
- If A and B are disjoint subsets of S, then $P(A \cup B) = P(A) + P(B)$.
- For any a and b in S, $P(\{a\}) = P(\{b\})$.

Notice that if S were finite, we would just have a uniform finite probability space.

(1) Show that for any positive real number x, there is a positive natural number n such that $\frac{1}{n} < x$. *Hint: Use Fact 9.32 from the notes.*

(2) Show that for any $a \in S$, $P(\{a\}) = 0$.

Now assume that $S = [0, 1]$. For a set $A \subseteq \mathbb{R}$ and a real number x, define $A + x := \{y \in \mathbb{R} \mid y = a + x \text{ for some } a \in A\}$. Assume the function P satisfies the following additional properties:

- If A_0, A_1, A_2, ... is a countable family of events such that $P(A_n) = 0$ for each $n \in \mathbb{N}$, then $P(\bigcup_{n \in \mathbb{N}} A_n) = 0$. (*So a "small" union of very unlikely event is still very unlikely.*)
- If A is an event, x is a real number and $A + x \subseteq S$, then $P(A + x) = P(A)$ (*So a "translate" of A has the same probability as A.*)

You should think of these axioms as trying to express the process of choosing elements of $[0, 1]$ uniformly at random. However, you will now show that there *cannot* exist a function P satisfying those very reasonable axioms!

Recall from assignment 5 that the relation E on $[0, 1]$ defined by xEy if and only if $x - y$ is rational is an equivalence relation. We now build a set $X \subseteq [0, 1]$ such that for each $x, y \in X$, if $[x]_E = [y]_E$, then $x = y$, and for any $y \in \mathbb{R}$, there is $x \in X$ such that xEy (you may take it for granted that such a set exists. Intuitively, we just look at the set $[0, 1]/E$ of all equivalence classes, pick exactly one element out of each class, and put it in X).

We will show that the event X cannot be assigned any probability, which will contradict our list of axioms. Recall that by assumption, $P([0, 1]) = P(S) = 1$.

(3) Show that for distinct rational numbers r and r', $(X + r) \cap (X + r') = \emptyset$.

(4) Show that $[0, 1]$ is the union of all sets of the form $(X + r) \cap [0, 1]$ for r a rational number. Conclude that X is uncountable.

(5) Show that if $P(X) > 0$, then $P([0, 1]) > 1$.

(6) Show that if $P(X) = 0$, then $P([0, 1]) = 0$.