CONCEPTS OF MATHEMATICS, SUMMER 1 2014
ASSIGNMENT 4

Due Tuesday, June 3 at the beginning of class. Make sure to include your name, Andrew ID, *and the list of your collaborators* (if any) with your assignment. You may discuss problems with others, but you may *not* keep a written record of your discussions. Please refer to the syllabus for details.

Problem 1 (20 points)

(1) Write the following sets using set-builder notation. This means your answer should be of the form $\{x \in A \mid p(x)\}$ for A a set and p a propositional function (that need not be written in symbols). For example, the set of non-negative real numbers is $\{x \in \mathbb{R} \mid x \text{ is non-negative}\}$ or alternatively $\{x \in \mathbb{R} \mid x \geq 0\}$.
   (a) The set of nonzero real numbers.
   (b) The set of irrational numbers.
   (c) The set of real numbers that are the square root of a natural number.
   (d) The set of subsets of reals that do not contain 0 but contain 1.

(2) (a) Compute $\mathcal{P}(\{4, 9, 13\})$.
   (b) Explain why $\emptyset \neq \mathcal{P}(\emptyset)$.
   (c) Compute $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset)))$.
   (d) Assume A and B are sets. Show that $A \subseteq B$ if and only if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Problem 2 (20 points)

For sets A and B, we define the *difference* $A \backslash B$ (read "A minus B", or "A without B") to be the set $\{x \in A \mid x \notin B\}$. The *symmetric difference* $A \Delta B$ is the set $(A \backslash B) \cup (B \backslash A)$. Given a fixed set U (often called the *universe*), the *complement* A^c of a set $A \subseteq U$ is defined to be $U \backslash A$.

(1) Prove De Morgan's laws for sets: for a universe set U, and $A, B \subseteq U$, $(A \cap B)^c = A^c \cup B^c$, and $(A \cup B)^c = A^c \cap B^c$.
(2) Show that $A \Delta B = (A \cup B) \backslash (A \cap B)$.

For a and b real numbers, we write (a, b) for the set $\{x \in \mathbb{R} \mid a < x < b\}$ and $[a, b]$ for $\{x \in \mathbb{R} \mid a \leq x \leq b\}$. We similarly define $(a, b]$ and $[a, b)$. Note that if $a > b$, then $[a, b] = \emptyset$, and if $a \geq b$, then $(a, b) = \emptyset$.

(3) Compute the intersection of all sets of the form $(0, b)$, for b a positive real number.

Date: May 30, 2014.

(4) Compute the union of all sets of the form $[a, 1]$, for a a positive real number.

Problem 3 (20 points)

Prove that for all natural numbers m and n, $m + n$ is a natural number. (This is part of Fact 3.6. One can do a similar proof for multiplication, and with some work replace "natural numbers" by "integers").

Hint: Fix m and use induction on n.

Problem 4 (20 points)

(1) For each of the set of real numbers below, say whether it is inductive and prove your claim:
  (a) $\mathbb{Z}$.
  (b) $\mathbb{R}\backslash\mathbb{Q}$.
  (c) $(\mathbb{R}\backslash\mathbb{Q}) \cup \mathbb{N}$.
  (d) $\mathbb{R}\backslash\{\frac{1}{2}\}$.

(2) Using the definition of the natural numbers as the intersection of all inductive sets, show that if x is a real number such that $0 < x < 1$, then $x \notin \mathbb{N}$. *Hint:* Consider the set $\{y \in \mathbb{R} \mid y > x\} \cup \{0\}$.

Problem 5 (20 points)

(1) Fix a real number $r \neq 1$. Show that for any natural number n, $\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$.

(2) Explain why we assumed $r \neq 1$ in (1), and give a simple formula to compute the sum in case $r = 1$.

You can assume the following basic fact: for real numbers a, b, c, d with b and d nonzero, $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$.

Extra credit: Russel's paradox (20 points)

Show that there is not set of all sets, i.e. there is no set A such that $B \in A$ exactly if B is a set. *Hint:* Suppose that such an A exists and derive a contradiction as follows: consider the family S of sets in A that are *not* members of themselves and ask whether S is a member of itself[1].

[1] Closer to the real world, think about why there cannot be a book listing all non-self-referential books.