CONCEPTS OF MATHEMATICS, SUMMER 1 2014
ASSIGNMENT 5

Due Friday, June 6 at the beginning of class. Make sure to include your name, Andrew ID, *and the list of your collaborators* (if any) with your assignment. You may discuss problems with others, but you may *not* keep a written record of your discussions. Please refer to the syllabus for details.

Problem 1 (20 points): Dividing

Assume n, m, and k are integers. Prove or disprove:

(1) If k divides n then (k divides m if and only if k divides $n+m$).
(2) If k divides $n+m$, then either k divides n or k divides m.
(3) If k divides n, then k divides $n \cdot m$.
(4) If k divides $n \cdot m$, then either k divides n or k divides m.
(5) If m and n are coprime, then m and $m-n$ are coprime.

Problem 2 (20 points): The well-ordering property

Recall that a is a *minimal element* of $X \subseteq \mathbb{R}$ if $a \in X$ and for any $b \in X$, $a \leq b$. Show that any non-empty subset of $\mathbb{N}$ has a minimal element. *Hint: Prove by strong induction on n that any subset of $\mathbb{N}$ which contains n has a minimal element.*

Problem 3 (20 points): The poor man's prime factorization

Show that for every nonzero integer n there exists a unique natural number m and a unique odd integer k such that $n = 2^m k$. Don't forget to prove uniqueness! (For the problems in this assignment, you may use that if x is a real number, m and k are natural numbers, then $x^{m+k} = x^m \cdot x^k$, and if $m \leq k$ and $x \neq 0$, $x^{k-m} = \frac{x^k}{x^m}$)

Problem 4 (20 points): Coding ordered pairs with sets

(1) For objects a and b, define (a, b) to be the set $\{\{a\}, \{a, b\}\}$. Show that for any objects a, b, c, d, if $(a, b) = (c, d)$, then $a = c$ and $b = d$.
(2) Prove or disprove:
    (a) For all sets A, B, C, D: $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$.
    (b) For all sets A, B, C, D: $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Date: June 4, 2014.

PROBLEM 5 (20 POINTS): ON EQUIVALENCE RELATIONS

(1) Give an example of a set A and a relation on A which is reflexive, symmetric, but not transitive.
(2) Show that the relation E on the set $\mathbb{R}$ defined by xEy if and only if $x - y$ is rational is an equivalence relation. Is this still true if we replace "rational" by "irrational"?
(3) Given an equivalence relation E on a set A and $a \in A$, the *equivalence class* $[a]_E$ *of* a is defined to be the set $\{b \in A \mid bEa\}$. Write A/E for the set of equivalence classes of E (in symbols, $A/E = \{x \in \mathcal{P}(A) \mid x = [a]_E \text{ for some } a \in A\}$). Show that for any a and b in A:
   (a) $a \in [a]_E$.
   (b) If aEb, then $[a]_E = [b]_E$, and if $\neg(aEb)$, then $[a]_E \cap [b]_E = \emptyset$.
   (c) A is the union of all sets of the form $[c]_E$ for $c \in A$.
(4) Given an equivalence relation E on a set A, show that there exists a set B and a function $f : A \to B$ such that for all $a, b \in A$, aEb if and only if $f(a) = f(b)$. *Hint: take* $B = A/E$.

EXTRA CREDIT (20 POINTS): THE FIBONACCI SEQUENCE

For n a natural number, define the nth *Fibonacci number* a_n inductively by:

- $a_0 = 0$.
- $a_1 = 1$.
- $a_n = a_{n-1} + a_{n-2}$ if $n \geq 2$.

(1) Compute explicitly a_0, a_1, a_2, ..., a_8.
(2) Show that $a_n = \frac{\phi^n - \psi^n}{\sqrt{5}}$, where $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$. *Hint: First show that* $\phi^2 = \phi + 1$ *and* $\psi^2 = \psi + 1$.