CONCEPTS OF MATHEMATICS, SUMMER 1 2014
ASSIGNMENT 6

Due Friday, June 13 at the beginning of class. Make sure to include your name, Andrew ID, *and the list of your collaborators* (if any) with your assignment. You may discuss problems with others, but you may *not* keep a written record of your discussions. Please refer to the syllabus for details.

Problem 1 (20 points)

Assume $f : A \to B$ and $g : B \to C$ are functions.

(1) Assume f and g are injective. Show that $g \circ f$ is injective.
(2) Assume f and g are surjective. Show that $g \circ f$ is surjective.
(3) Assume $g \circ f$ is a bijection. Is f injective? Is g injective?
(4) Assume $g \circ f$ is a bijection. Is f surjective? Is g surjective?

Problem 2 (20 points)

(1) Assume $f : A \to B$ is a bijection. Prove that f^{-1} is a bijection.
(2) Assume $f : A \to B$ is an injection. Show that for any set $C \subseteq A$, C is equipotent to its image $f[C]$.
(3) Assume U is a set. Show that the relation E on $\mathcal{P}(U)$ defined by AEB if and only if A is equipotent to B is an equivalence relation.
(4) Show that if A is a set and n and m are natural numbers such that A is equipotent to both $[n]$ and $[m]$, then $n = m$. Do not use the definition of $|A|$.

Problem 3 (20 points)

Show that:

(1) If n is a natural number and $A \subseteq [n]$. Then A is finite and $|A| \leq n$ with equality if and only if $A = [n]$. *Hint: use induction on n.*
(2) For natural numbers n and m, if $f : [n] \to [m]$ is an injection, then $n \leq m$. *Hint: use (1).*
(3) If $f : A \to B$ is an injection and B is finite, then A is finite. Conclude that if $A \subseteq B$ and B is finite, then A is finite, and if $A \subseteq B$ and A is infinite, then B is infinite. *Hint: use (1).*
(4) If A and B are countable, then $A \times B$ is countable.

Date: June 6, 2014.

Problem 4 (20 points): Size of the power set

(1) Fix a natural number n. Formally, a binary *n-tuple* s is a function from $[n]$ to $\{0,1\}$. You can see s as a list $s(1)$, $s(2)$, $\ldots$, $s(n)$ of n zeroes or ones. Show that there is a bijection from $\mathcal{P}([n])$ to the set of binary n-tuples (that is, one can "code" subsets of $[n]$ using n-tuples of zeroes and ones). *Hint: let the ith element of the tuple be 1 if i is in the set and* 0 *otherwise.* Use this result to explain informally why $|\mathcal{P}([n])| = 2^n$.

(2) For the rest of this problem, assume A is a set. Show that there is an injection $f : A \to \mathcal{P}(A)$.

(3) Write ${}^A\{0,1\}$ for the set of functions from A to $\{0,1\}$. Generalize (1) by showing that there is a bijection from $\mathcal{P}(A)$ to ${}^A\{0,1\}$.

(4) Write ${}^AA$ for the set of functions from A to A and assume A has at least two elements. Show that there is an injection from $\mathcal{P}(A)$ to ${}^AA$. Is this still true if A has one element? What about no elements?

Problem 5 (20 points)

(1) For each of the following four conditions, give an example of a function from $\mathbb{N}$ to $\mathbb{N}$ which satisfies it: bijective and not the identity, injective and not surjective, surjective and not injective, not surjective and not injective.

(2) We say s is a *bit string* if it is a binary n-tuple for some natural number n. Show that the set of all bit strings is countable.

(3) Show that the set $\{x \in \mathbb{R} \mid x = a + b\sqrt{42} \text{ for some } a, b \in \mathbb{Q}\}$ is countable.

(4) Show that the set of irrational numbers is uncountable.

Extra credit (20 points): Cantor's theorem and uncomputability

(1) Prove Cantor's theorem: given a set A, there is no surjection from A to $\mathcal{P}(A)$ (so in this sense, the cardinality of $\mathcal{P}(A)$ is strictly larger than that of A). *Hint: Assume for a contradiction there is such a surjection $F : A \to \mathcal{P}(A)$, and consider the set $\{a \in A \mid a \notin F(a)\}$.*

(2) At the bottom, computers deal with bit strings: finite strings of 0s and 1s (i.e. binary n-tuple for some n). We think of a computer program simply as a bit string and assume that each computer program P computes a function f_P that takes as input a bit string and outputs another bit string. For example, one can write a (very rudimentary) calculator program P for which f_P would take as input a bit string coding two natural number n and m and output a bit string coding $n+m$. Show that there is a function from the set of bit strings to the set of bit strings that no computer program can compute. *Hint: How many programs are there? How many functions are there?*