CONCEPTS OF MATHEMATICS, SUMMER 1 2014
ASSIGNMENT 7

Due Tuesday, June 17 at the beginning of class. Make sure to include your name, Andrew ID, *and the list of your collaborators* (if any) with your assignment. You may discuss problems with others, but you may *not* keep a written record of your discussions. Please refer to the syllabus for details.

Problem 1 (20 points)

(1) Prove the rule of sum for two sets: For any disjoint finite sets A and B, $|A \cup B| = |A| + |B|$. *Hint: Assume A has size n, B has size m, and show $A \cup B$ and $[n+m]$ are equipotent.*
(2) Prove the inclusion-exclusion principle: For any finite sets A and B, $|A \cup B| = |A| + |B| - |A \cap B|$. *Hint: Decompose the union into disjoint sets and use the rule of sum.*

Problem 2 (20 points)

Assume n, m, and k are natural numbers. Give a combinatorial proof of the following identities by "counting in two ways".

(1) $2^n = \sum_{i=0}^{n} \binom{n}{i}$ (also explain how this follows from the binomial theorem).
(2) For $n \geq 1$, $\sum_{i=0}^{n} i\binom{n}{i} = n2^{n-1}$.
(3) $\sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k}$.

Give an algebraic proof that for any natural numbers n and k:

(4) $\sum_{i=0}^{n} \binom{i}{k} = \binom{n+1}{k+1}$

Problem 3 (20 points)

For this problem and the next, an answer given as some expression involving binomial coefficients and factorials (like $123!\binom{42}{21}$) is fine, no need to compute a final number. Also recall that poker hands are considered to be unordered (i.e. we think of them as sets).

(1) How many poker hands are there that contain two pairs and only two suits?
(2) How many poker hands are there that contain either exactly three jacks or exactly two spades (or both)?
(3) How many poker hands are there that contain five distinct ranks, including a two and a three?
(4) How many poker hands are there that contain at most two different suits and five distinct ranks?

Problem 4 (20 points)

(1) How many different arrangements can one make with the letters of INFINITY (for example, FINITYIN would be one such arrangement)?
(2) (a) How many *strictly positive* natural number solutions does the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 123$ have? *Hint: Do a change of variables to reduce this back to counting the number of natural number solutions.*

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(b) How many natural number solutions does the equation $x_1+x_2+x_3+x_4+x_5 \leq 123$ have? Your answer should not include long sums. *Hint: Add a variable.*

(3) Assume m and n are natural numbers, and we are given m different objects and n different boxes. In how many different ways can we put objects into boxes (without restrictions)? In how many different ways can we put objects into boxes so that two distinct objects are never in the same box? We assume that both boxes and objects are distinguishable so for example if $n = m = 2$, putting object 1 in box 1 and object 2 in box 2 is not the same as putting object 1 in box 2 and object 2 in box 1.

(4) For $n \geq 7$ a natural number, how many binary n-tuples are there that either begin with 01, end with 1110, or have an even number of 1s? *Warning: as always in mathematics, "or" is not the same as "exclusive or".*

Problem 5 (20 points)

(1) We arrange the natural numbers from 1 to 10 on a circle in some arbitrary order. Show that as we go around the circle clockwise, there must be three consecutive (in the order of the circle) numbers whose sum is at least 17.

(2) Show that at a party with $n \geq 2$ students, there are two students who know the same number of people (we assume knowledge is symmetric).

Extra credit (20 points): Infinity crashes the party

Show that in an infinite party with countably many students, there exists a countable set S of students such that either all students in S know each other, or all students in S do not know each other. We make the standard assumption that knowledge is symmetric.

Hint: First prove a suitable infinite version of the pigeonhole principle. Then given a student x_0, argue that it either knows countably many other students, or does not know countably many other students. Pick x_1 among those students, and continue in this way to build a sequence x_0, x_1, x_2, ... Such that for each i, x_i either knows all the students in $x_{i+1}, x_{i+2}, \ldots$, or does not know any of them. Conclude by using your (infinite) pigeonhole principle one last time.