CONCEPTS OF MATHEMATICS, SUMMER 1 2014
ASSIGNMENT 8

Due Friday, June 20 at the beginning of class. Make sure to include your name, Andrew ID, *and the list of your collaborators* (if any) with your assignment. You may discuss problems with others, but you may *not* keep a written record of your discussions. Please refer to the syllabus for details.

Problem 1 (20 points)

Prove Theorem 10.11 in the lecture notes: For any integers m, n, and k:

(1) $\gcd(n, m) = \gcd(m, n)$.
(2) $\gcd(n, m) = \gcd(\pm n, \pm m)$ (i.e. equality holds regardless of the choice of sign).
(3) $\gcd(n, m) = \gcd(n, m + kn)$.
(4) $\gcd(n, m) = 1$ if and only if n and m are coprime.
(5) $\gcd(n, m) = 0$ if and only if $n = m = 0$.
(6) If n divides m, $\gcd(n, m) = |n|$. In particular, $\gcd(n, 0) = |n|$.
(7) If $n = ka$, $m = kb$, and a and b are coprime, then $\gcd(n, m) = |k|$. *Warning: you cannot use that the gcd can be computed from the prime factorization, as it is a consequence of that problem.*

Problem 2 (20 points)

Prove Theorem 10.15 in the lecture notes: For integers m, n, and k, the following are equivalent (i.e. (A) holds if and only if (B) holds):

- (A) $\gcd(n, m)$ divides k.
- (B) There exists integers a and b such that $k = am + bn$.

Problem 3 (20 points)

Assume p is a prime number.

(1) Show that p divides $\binom{p}{k}$ for all natural numbers k with $1 \leq k \leq p - 1$.
(2) Show that for any natural number n, p divides $n^p - n$. *Hint: induction.*

Problem 4 (20 points)

Use the Euclidean algorithm to find $\gcd(123, 321)$, as well as two integers a and b such that $123a + 321b = \gcd(123, 321)$. Show all your work.

Problem 5 (20 points)

Assume n is a natural number. Show that there exists a natural number m such that none of $m, m+1, \ldots, m+n$ are prime. *Hint: Find m such that m is a multiple of 2, $m+1$ is a multiple of 3, $m+2$ a multiple of 4, etc.*

Date: June 17, 2014.

Extra credit (20 points): Perfect numbers

We start with a couple of definitions and fun facts. A natural number n is called *perfect* if it is equal to the sum of its proper (i.e. not equal to n) natural number divisors. For example, 6 is perfect: the natural number divisors of 6 are 1, 2, and 3, and $6 = 1 + 2 + 3$. $28 = 1 + 2 + 4 + 7 + 14$ is also perfect (it is unknown whether there are infinitely many perfect numbers, or whether there are any odd perfect numbers at all). A *Mersenne prime* is a prime of the form $2^n - 1$ for n a natural number. For example, $7 = 2^3 - 1$ is a Mersenne prime (It is also unknown whether there are infinitely many Mersenne primes. The largest known Mersenne prime is $2^{57885161} - 1$, found by a distributed effort over the internet).

Show that if $2^n - 1$ is a Mersenne prime, then $2^{n-1}(2^n - 1)$ is perfect.