CONCEPTS OF MATHEMATICS, SUMMER 1 2014
ADDITIONAL EXERCISES FOR WEEK 1

(1) Recall that Goldbach's conjecture says that every even natural number larger than 2 is the sum of two primes (nobody knows whether it is true or not). Determine the truth value of the following propositions:
   (a) $0 + 0 = 0$ if and only if $0 = 2$.
   (b) If Goldbach's conjecture is true, then Goldbach's conjecture is true.
   (c) Goldbach's conjecture is true or $\sqrt{2}$ is irrational.
   (d) (Goldbach's conjecture is true or $\sqrt{2}$ is irrational) if and only if Goldbach's conjecture is true.
   (e) (Goldbach's conjecture is true or $\sqrt{2}$ is irrational) if and only if Goldbach's conjecture is false.

(2) Prove or disprove: for all real numbers x, y, and z:
   (a) $|x - y| \geq |x| - |y|$.
   (b) $(x + y)^2 \leq x^2 + y^2$.
   (c) $x + y + z \leq \left(\frac{x+y+z}{3}\right)^3$.
   (d) $|x + y| = |x| + |y|$ if and only if both x and y are non-negative.

(3) What is wrong with the following "proof" that $1 = -1$?
   Assume x and y are real numbers with $y = x + 1$. Then $(x - y)^2 = x^2 - 2xy + y^2$, so

$$x-y = \sqrt{x^2 - 2xy + y^2} = \sqrt{x^2 - 2x(x + 1) + (x + 1)^2} = \sqrt{x^2 - 2x^2 - 2x + x^2 + 2x + 1} = 1$$

   so $x + 1 = y = x - 1$, so $1 = -1$.

(4) A *tautology* is a proposition that is logically equivalent to T. A *contradiction* is a proposition that is logically equivalent to F. Assume p and q are propositions. Show:
   (a) $p \vee \neg p$ is a tautology.
   (b) $p \wedge \neg p$ is a contradiction.
   (c) $p \to p$ is a tautology.
   (d) $p \leftrightarrow \neg p$ is a contradiction.
   (e) $(p \leftrightarrow q) \to (p \to q)$ is a tautology.
   (f) $F \to q$ is a tautology.

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(g) If P and Q are propositions involving p and q, then $P \equiv Q$ precisely when $P \leftrightarrow Q$ is a tautology.

(5) Assume p is a proposition. Show that $p \to F \equiv \neg p$.

(6) Assume $p(x)$ is a propositional function. Write the statement "There exists a *unique* x such that $p(x)$" using only $\exists$, $\forall$, $=$, $\vee$, $\neg$, $\wedge$, $\to$, and $\leftrightarrow$.

(7) Prove the principle of modus ponens: $((p \to q) \wedge p) \to q$ is always true.

(8) Justify the assertion that when proving p, one can always assume $\neg p$ for free by showing that $(\neg p \to p) \to p$ is a tautology.

(9) Write the following propositions in symbols. Assume the domain of discourse consists of "all persons having ever lived".
  (a) All mathematicians are liars. (You can use the propositional function $M(x)$ to mean "x is a mathematician, and $L(x)$ to mean "x is a liar").
  (b) Gauss is a mathematician. (Write g for the object denoting Gauss).
  (c) Some mathematicians are not liars.
  (d) For every mathematician, there is a different mathematician which has a different philosophy (you can write $P(x, y)$ to mean that x and y share the same philosophy).
  (e) For every mathematician, there is a different mathematician which has a different philosophy (you can write $P(x, y)$ to mean that x and y share the same philosophy).
  (f) There is a mathematician whose philosophy is different from every other mathematician.

(10) Assume you know:
  (a) Every mathematician is a liar.
  (b) Gauss is a mathematician.

  Write these in symbols and prove in details that this implies that Gauss is a liar.