CONCEPTS OF MATHEMATICS, SUMMER 1 2014
ADDITIONAL EXERCISES FOR WEEK 2 (UPDATED)

(1) Do a proof by case of the triangle inequality.

(2) Inhabitants of wonderland are of two different sorts: One sort always lies, and another sort always tells the truth. Let A and B be two different inhabitants of wonderland. A says "B always tell the truth". B says "We are of two different sorts". Determine the sorts of A and B.

(3) Prove the following properties of summation.

Assume $f(i), g(i)$ are expressions depending on i, $h(i,j)$ is an expression depending on both i and j, n, m are a natural numbers, and c is a real number. Then:

(a) $\sum_{i=1}^{n} f(i) + g(i) = (\sum_{i=1}^{n} f(i)) + (\sum_{i=1}^{n} g(i))$.
(b) $\sum_{i=1}^{n} cf(i) = c \sum_{i=1}^{n} f(i)$.
(c) $|\sum_{i=1}^{n} f(i)| \leq \sum_{i=1}^{n} |f(i)|$.
(d) $(\sum_{i=1}^{n} f(i))(\sum_{j=1}^{m} g(j)) = \sum_{i=1}^{n} \sum_{j=1}^{m} f(i)g(j)$.
(e) $\sum_{i=1}^{n} \sum_{j=1}^{m} h(i,j) = \sum_{j=1}^{m} \sum_{i=1}^{n} h(i,j)$.

Hint: Use induction

(4) What is wrong with the following "proof" that all books have the same title?

We use the principle of mathematical induction on the propositional function $p(x)$ saying "any set of x books have the same title". For the base case, $p(0)$ and $p(1)$ clearly holds. For the inductive step, assume any set of n books have the same title. X is nonempty, so pick a book b in X. The set $X' := X \backslash \{b\}$ has n elements, so by the induction hypothesis, all books in X' have the same title, so all elements of X have the same title except perhaps b. Take an element a in X distinct from b (this is possible since we can assume $n \geq 1$), and consider now the set $X'' := X \backslash \{a\}$. Since a is distinct from b, b is still in X'', and by the induction hypothesis all books in X'' have the same title, so in particular b also has the same title as the others. Thus all books in X have the same title.

By the principle of mathematical induction, all books have the same title.

Date: June 8, 2014.

(5) Show the following elementary properties of x^n: for real numbers x and y, and natural numbers n and m:
  (a) $1^n = 1$.
  (b) $0^n = 0$ if $n > 0$ and $0^0 = 1$.
  (c) $x^n \cdot x^m = x^{n+m}$.
  (d) $x^n \cdot y^n = (x \cdot y)^n$
  (e) $(x^n)^m = x^{nm}$.
  (f) If x is non-negative, then x^n is non-negative.
  (g) If x is negative, then x^n is negative if n is odd and positive otherwise.
  (h) If $x \geq 2$, $n < x^n$.

(6) Assume n is a natural number and $p_0, \dots, p_n$ are prime numbers. Show there exists a prime number q with $q \neq p_i$ for any $0 \leq i \leq n$. *Hint: Consider the number* $p_0 \cdot \dots \cdot p_n + 1$.

(7) The US government suddenly issues new \$7 and \$9 bills and declares that everything has to be paid using these bills. Compute all the amounts that can be paid.

(8) Show that for any integers n and m with m nonzero, there exists unique integers k and r such that $n = mk + r$ and $0 \leq r < |m|$. We call k the *quotient* and r the *remainder* of the division of n by m.

(9) We consider an 8×8 chessboard and remove two diagonaly opposite corners. Show that it is impossible to cover such a chessboard with dominoes (we assume each domino takes up exactly two squares of the board). *Hint: assume it is possible and recall that a chessboard has black and white squares..*

(10) n fully rational prisoners are in a room. Each prisoner wears either a white or black hat, and at least one wears a white hat. Each prisoner can see the color of the other hats but not the color of his/her own's. The prisonners are told that the light will be switched off, and that they must either all stay, or exactly those with a white hat should leave. If they all stay, the experiments is repeated, until the prisoners die of thirst if necessary. If only those with a white hat leave, all the prisoners are instantly freed. If something else happens, all prisoners are condemned to live the rest of their lives in jail making sandwiches for their jailers. Describe what happens. *Hint: induction.*

(11) Show that $\sqrt{3}$ is irrational.

(12) For positive real numbers x, y, z, assume $x > z$ and $y^2 = xz$. Show that $x > y > z$. (This is Problem 2.17 in Martin V. Day's notes). *Hint: suppose not, and consider the two natural cases.*

(13) Show that for any positive natural number n, $n - 1$ is a natural number. *Hint: induction*

(14) Show that for integers n and m, $n + m$ and $n \cdot m$ are integers. *Hint: first show it for natural numbers.*