CONCEPTS OF MATHEMATICS, SUMMER 1 2014
ADDITIONAL EXERCISES FOR WEEK 3

(1) Assume $f : A \to B$ is a function. Assume A', A'' are subsets of A and B', B'' are subsets of B. Prove or disprove:
  (a) If $A' \subseteq A''$, then $f[A'] \subseteq f[A'']$.
  (b) If $B' \subseteq B''$, then $f^{-1}[B'] \subseteq f^{-1}[B'']$.
  (c) $f[A' \cup A''] = f[A'] \cup f[A'']$.
  (d) $f[A' \cap A''] = f[A'] \cap f[A'']$.
  (e) $f^{-1}[B' \cup B''] = f^{-1}[B'] \cup f^{-1}[B'']$.
  (f) $f^{-1}[B' \cap B''] = f^{-1}[B'] \cap f^{-1}[B'']$.

(2) If $f : A \to B$ is a bijection and $C \subseteq B$, then $f[f^{-1}[C]] = C$. Is this still true if we do not assume that f is a bijection?

(3) A is infinite if and only if it is equipotent to one of its proper subset.

(4) There is a surjection from A to B if and only if there is an injection from B to A.

(5) A function $f : \mathbb{R} \to \mathbb{R}$ is called *increasing* if for any $x < y$, $f(x) < f(y)$. It is called *decreasing* if for any $x < y$, $f(x) > f(y)$. It is called monotone if it is either increasing or decreasing. Show that monotone functions are injective.

(6) For sets A and B, we write $A \preceq B$ if there is an injection from A to B, and $A \approx B$ if A is equipotent to B. We write $A \prec B$ if $A \preceq B$ and $A \not\approx B$.

   Show that:
  (a) For natural numbers n and m, $[n] \preceq [m]$ if and only if $n \leq m$, $[n] \approx [m]$ if and only if $n = m$, and $[n] \prec [m]$ if and only if $n < m$.
  (b) If $A \subseteq B$, then $A \preceq B$.
  (c) If $A \approx A'$ and $B \approx B'$, then $A \preceq B$ if and only if $A' \preceq B'$.
  (d) If $A \approx B$, then $A \preceq B$ and $B \preceq A$.
  (e) If $A \preceq B$ and $B \preceq C$, then $A \preceq C$.
  (f) $A \prec \mathcal{P}(A)$.

(7) Give an explicit formula for the bijection from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$ obtained by listing the pairs whose sum is 0, then the pairs whose sum is 1, and so on. For pairs of the same sum, we list

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the pairs by decreasing order of first component, i.e. first we list $(n, 0)$, then $(n-1, 1)$, etc.

(8) (Very hard: this is known as the Cantor-Schröder-Bernstein theorem) Assume there is an injection from A to B and an injection from B to A. Show that there is a bijection from A to B. Using the terminology of the previous problem, if $A \preceq B$ and $B \preceq A$, then $A \approx B$.

Conclude that if $A \preceq B \preceq C$ and $A \approx C$, then $A \approx B$.

(9) A real number x is called *algebraic* if it is the root of a nonzero polynomial with rational coefficients, i.e. it satisfies an equation of the form $a_0 + a_1 x + a_2 x^2 + ... + a_n x^n = 0$, where $a_0 \neq 0$ and a_i is a rational number for all $i \leq n$. For example, $\frac{5}{42}$ is algebraic, as it is the root of the polynomial $x - \frac{5}{42}$, and $\sqrt{2}$ is algebraic, as it is a root of the polynomial $x^2 - 2 = 0$.

Prove that the set of algebraic numbers is countable. Conclude that there must exist a real number which is not algebraic. Such a number is called *transcendental*.

(10) A *partition* of a set X is a non-empty family S of non-empty subsets of X such that:
   (a) If $A, B \in S$, then either $A = B$, or $A \cap B = \emptyset$.
   (b) $\bigcup_{A \in S} A = X$.

Prove that given an equivalence relation E on X, the set S of equivalence classes of E is a partition of X. Conversely, show that for any partition S of X, there is a *unique* equivalence relation E whose set of equivalence classes is S.

(11) Prove that any natural number ≥ 2 can be written as a sum of prime numbers. Is this decomposition unique?

(12) Show that if A and B are disjoint finite sets, then $|A \cup B| = |A| + |B|$.

(13) Recall that the Fibonacci numbers are defined inductively by $a_0 = 0$, $a_1 = 1$, $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. Show that every natural number $n \geq 1$ can be written uniquely as a sum of distinct non-consecutive Fibonacci numbers (omitting a_0 from the decomposition).