CONCEPTS OF MATHEMATICS, SUMMER 1 2014
ADDITIONAL EXERCISES FOR WEEK 4

(1) For real numbers x and y, we say $x \equiv y \bmod 1$ if $x - y$ is an integer. Notice that if we restrict x and y to be integers, we get back the mod 1 relation defined in the notes.
   (a) Show that this defines an equivalence relation on the reals.
   (b) Show that $x \equiv \langle x \rangle \bmod 1$ (recall that $\langle x \rangle$ denotes the fractional part of x).
   (c) Show that if $x_0 \equiv x_1 \bmod 1$ and $y_0 \equiv y_1 \bmod 1$, then $x_0 + y_0 \equiv x_1 + y_1 \bmod 1$.
   (d) Show that if $x_0 \equiv x_1 \bmod 1$ and $c \in \mathbb{Z}$, then $cx_0 \equiv cx_1 \bmod 1$.
   (e) Now assume x and α are real numbers, i and j are integers. Assume you know that $|\langle ix \rangle - \langle jx \rangle| < \alpha$. Show that for some integer k, $|(i-j)x - k| < \alpha$.

(2) Show that for any positive real number x, there exists a positive natural number n such that $\frac{1}{n} < x$. Conclude by computing $\bigcap_{n=1}^{\infty}(0, \frac{1}{n})$.

(3) Using Dedekind's approximation theorem, show that for any real numbers $x < y$, there exists a rational number r with $x < r < y$.

(4) (Hard) Using the previous exercise, conclude that any set A of reals which satisties the well-ordering principle (i.e. every non-empty subset of A has a minimal element) is at most countable. *Hint: Find an injection of A into the rationals.*

(5) Show that for any natural number n, there exists a sequence of n^2 distinct real numbers which does not contain any monotone subsequence of length $n + 1$.

(6) Show by an example that it is not necessarily true that in a party with five students, there exists three students that either all know each other, or all do not know each other.

(7) Show that any subset of $[2n]$ of size $n + 1$ contains two coprime elements. Show that this is no longer true if we replace $n + 1$ by n.

(8) Prove the rule of product: for finite sets $A_0, A_1, \ldots, A_n$, $|A_0 \times A_1 \times \ldots \times A_n| = \prod_{i=0}^{n} |A_i|$.

(9) Show that for any natural numbers n and k, $\binom{n}{k} = \binom{n}{n-k}$ (give both a combinatorial and an algebraic proof).

(10) Prove that for $0 \leq k \leq n$, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ by induction using Pascal's formula.

(11) If S is a set of $n+1$ numbers in $[2n]$, then there exists distinct m, k in S such that m divides k. Show that this is no longer true if we replace $n + 1$ by n. *Hint: use the pigeonhole principle and the fact that every nonzero integer can be written as $2^m k$ with k odd...*

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(12) (Hard: will become easier once we study modular arithmetic) Assume p is a prime, and a is an integer. Show that if p does not divide a, then there exists an integer b such that p divides $ab - 1$.
(13) Make up your own set of poker hands and try to determine its size.
(14) Prove the following infinite versions of the pigeonhole principle:
  (a) If A is infinite, B is finite, $f : A \to B$, then there exists $b \in B$ such that $f^{-1}[\{b\}]$ is infinite.
  (b) If A is uncountable, B is at most countable, $f : A \to B$, then there exists $b \in B$ such that $f^{-1}[\{b\}]$ is uncountable.
(15) (Very hard: this is called the finite Ramsey theorem): For any natural number k, there exists a natural number n (potentially much bigger than k) such that in any party with n students, there is a group of k students that either all know each other, or all do not know each other. (So we have seen in class that if $k = 3$, then $n = 6$ is enough).
(16) Given five integer points in the plane (i.e. five distinct elements of $\mathbb{Z} \times \mathbb{Z}$), the midpoint of the segment joining two of them is also an integer point. Show that this is no longer true if we replace five by four.
(17) Assume you start at point $(0, 0)$ of a grid and want to reach the point (m, n) for m, n natural numbers. You are only allowed to go one step up or one step right (e.g. a possible path to $(2, 1)$ is $(0, 0), (0, 1), (1, 1), (2, 1)$). How many possible paths are there?
(18) Prove that for any real numbers x and y, and any natural number n, $x^n - y^n = (x - y) \sum_{i=0}^{n-1} x^i y^{n-i-1}$.
(19) (Hard) Prove that for any real number C, and any natural number k, there exists a natural number N such that $Cn^k \leq 2^n$ for all natural numbers $n \geq N$ (that is, Cn^k is smaller than 2^n for "large enough" values of n). *Hint: use induction on k, and use induction on n inside the induction on k. The binomial theorem might come in handy.*