# CONCEPTS OF MATHEMATICS, SUMMER 1 2014
# ADDITIONAL EXERCISES FOR WEEK 6

(1) We say the events $A_1$, $A_2$, ..., $A_n$ are *independent* if for any $S \subseteq [n]$, $P(\bigcap_{i \in S} A_i) = \prod_{i \in S} P(A_i)$. Give an example of a probability space with three events that are pairwise independent but not independent. *Hint: Look at whether the sum of three numbers is even or odd.*

(2) What is the probability that a randomly selected number in $[n]$ is a multiple of 2 or 3?

(3) A binary $n$-tuple is generated by tossing $n$ fair coins independently. What is the probability that such a tuple has at least two 1s? What is the probability that it has an even number of 1s?

(4) Alice and Bob are communicating over a crappy wireless channel. Alice only sends either 0 or 1s to Bob. Whenever she sends a 0, there is a 15% chance that channel noise turns it into a 1. Whenever she sends a 1, there is a 25% chance that it gets turned into a 0. Assume that Alice sends a 0 with probability 1/2. Given that Bob received a zero, what is the probability that Alice sent a zero? Same question with "zero" replaced by "one". Now assume that the bits Alice sends are all independent. What is the probability that no error happens if Alice sends $n$ bits? What is the probability that at most two bits out of $n$ get switched?

(5) Assume that 90% of email messages are spams, and that 80% of all spams contain the word "pill" whereas only 1% of nonspams contain that word. Given that an email contains the word "pill", what is the probability that it is a spam?

(6) Prove or disprove. For events $A$ and $B$, $P(A \backslash B) = P(A) - P(B)$.

(7) Assume $S = A_1 \cup A_2 \cup \ldots \cup A_n$, where the $A_i$s are pairwise disjoint. Show that for some $i \in [n]$, $P(A_i) \geq \frac{1}{n}$.

(8) Fix a natural number $n$. Alice chooses a natural number $k \in [n]$ at random, and then throws a $k$-sided fair die. What is the probability to roll a number $\geq \frac{n}{2}$? *Hint: condition on the value of $k$.*

(9) Three prisoners A, B and C are condemned to an entire year of potato peeling, but their jailer decides to choose one of them uniformly at random to pardon. The jailer does not want them to know (until the last minute) who exactly has been pardoned. A asks the jailer for the name of one of the two other non-pardoned prisoners. More precisely, the jailer is asked to do the following:
   (a) If A has been pardoned, give the name of either B or C uniformly at random.
   (b) If B has been pardoned, give the name of C. If C has been pardoned, give the name of B.

   Given all this, what is the probability that A has been pardoned?

*Date*: June 22, 2014.

(10) The score for your final exam will be a natural number in $\{0\} \cup [120]$. Assuming the scores are uniformly distributed at random, what is the probability that at least two of the 35 students in this class will share the same score? What is the probability that at least two students will get a 120? What is the probability that exactly two students will get a 120? What is the probability that nobody will get a 120? What is the probability that at least one student will get a score which is a prime number? A multiple of 2? A multiple of 7? A multiple of 2 or a multiple of 3?

(11) Derive directly from the other axioms that a probability space must be non-empty.

(12) Show that in a uniform finite probability space, the probability of any event will be a rational number. Is this still true in a non-uniform space?

(13) A university is found to have accepted only 45% of the male undergraduate applicants, while 55% of the female applicants were accepted. Does this necessarily mean that male students were discriminated against?