CONCEPTS OF MATHEMATICS, SUMMER 1 2014
FACTS ABOUT THE REAL NUMBERS

Fact (Axioms of real numbers)**.** The real numbers are objects satisfying the following properties:

- (R_0) Among the reals, there are two distinguished elements, 0 and 1, with $0 \neq 1$. 0 and 1 have some special properties discussed below.
- (R_1) Binary operations $+$ and $\cdot$ are defined on the reals (they take two reals as input and produce one real as output).
- (R_2) Between any two reals x and y, one can ask whether $x < y$.
- Addition ($+$) satisfies the following properties: For all real numbers x, y, z:
  - (A_0) Associativity: $(x + y) + z = x + (y + z)$.
  - (A_1) Commutativity: $x + y = y + x$.
  - (A_2) Zero is the additive identity: $x + 0 = x$.
  - (A_3) Existence of inverse: There is always a unique real number w such that $x + w = 0$.
- Multiplication ($\cdot$) satisfies the following properties: For all real numbers x, y, z:
  - (M_0) Associativity: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
  - (M_1) Commutativity: $x \cdot y = y \cdot x$.
  - (M_2) One is the multiplicative identity: $x \cdot 1 = x$.
  - (M_3) Existence of inverse: If $x \neq 0$, there is a unique real number w such that $x \cdot w = 1$.
- Multiplication and addition interact as follows: For all real numbers x, y, z:
  - (D_0) Distributive law: $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$.
- The ordering ($<$) satisfies the following properties: For all real numbers x, y, z:
  - (O_0) Trichotomy: exactly one of the following is true: $0 < x$, $x = 0$, or $x < 0$.
  - (O_1) Closure under addition: If $0 < x$ and $0 < y$, then $0 < x + y$.
  - (O_2) Closure under multiplication: If $0 < x$ and $0 < y$, then $0 < x \cdot y$.
  - (O_3) If $x < y$, then $x + z < y + z$.

- (C_0) The completeness axiom (*will not be discussed in this course*): If F is a non-empty collection of real numbers and there is a real number x such that for all y in F, $y < x$, *then* one can choose x with the additional property that for any $x' < x$ there is y in F with $x' < y$.

Definition. For x a real number, the *negative* of x, $-x$, is the *unique* real number w such that $x + w = 0$ Similarly, define the *reciprocal*, x^{-1}, of a nonzero x to be the *unique* w such that $xw = 1$.

For real numbers x, y, we define $x - y$ to mean $x + (-y)$. Similarly, for y nonzero, we define x/y (also written $\frac{x}{y}$) to mean $x \cdot y^{-1}$.

Fact (Properties of addition and multiplication)**.** For all real numbers x, y, z, w:

- (F_0): $x \cdot 0 = 0$.
- (F_1): $-(xy) = (-x)y$.
- (F_2): $-x = (-1)x$.
- (F_3): $(-x)(-y) = xy$.
- (F_4): If $xy = 0$, then $x = 0$ or $y = 0$ (or both).
- (F_5): $(x + y)(z + w) = xz + xw + yz + yw$.

Fact (Properties of the ordering)**.** For all real numbers x, y, z, w:

- (F_6): Totality: Exactly one of $x < y$, $x = y$, $y < x$ always holds. Exactly one of $x \leq y$ or $y < x$ always holds.
- (F_7): Reflexivity: $x \leq x$.
- (F_8): Antisymmetry: If $x \leq y$ and $y \leq x$, then $x = y$.
- (F_9): Transitivity: If $x \leq y$ and $y \leq z$, then $x \leq z$. Similarly if $\leq$ is replaced by $<$.
- Interaction with addition and multiplication:
  - (F_{10}): $0 < 1$.
  - (F_{11}): If $x \leq y$ and $z \leq w$, then $x + z \leq y + w$. Similarly if $\leq$ is replaced by $<$.
  - (F_{12}): If $x \leq y$, then $-y \leq -x$. Similarly if $\leq$ is replaced by $<$.
  - (F_{13}): If $x \leq y$ and $0 \leq z$, then $xz \leq yz$.
  - (F_{14}): If $0 \leq x$ and $0 \leq y$, then $0 \leq xy$. Similarly if $\leq$ is replaced by $<$.
  - (F_{15}): $0 \leq x \cdot x$, and if $0 < x$ then $0 < x \cdot x$.
  - (F_{16}): If $0 < x$, then $0 < x^{-1}$.
  - (F_{17}): If $0 < x < y$, then $0 < y^{-1} < x^{-1}$.