CONCEPTS OF MATHEMATICS, SUMMER 1 2014
SAMPLE FINAL EXAM

First name: ______________________________
Last name: ______________________________
Andrew ID: ______________________________

You have one hour and twenty minutes to complete the exam. The exam should contain five problems and one extra credit question.

You may *not* use any notes or other documents, and you may *not* use any electronics such as a calculator or a mobile phone. You may use your brain and a pen.

Quote of the day: "There is the problem. Seek its solution. You can find it by pure reason, for in mathematics there is no ignorabimus." (David Hilbert).

Good luck!

Problem 1		20
Problem 2		20
Problem 3		30
Problem 4		15
Problem 5		15
Extra Credit		20
Total		100

Problem 1 (20 points)

State the following results/definitions discussed in class. Do *not* prove anything.

(1) What it means for two integers to be congruent modulo an integer n.
(2) The fundamental theorem of arithmetic.
(3) Bayes' theorem.
(4) The number of ways to choose k out of n elements when order matters and repetitions are permitted (for $k, n \in \mathbb{N}$).
(5) The definition of a surjection.

Date: June 27, 2014.

Problem 2 (20 points)

Say whether the following statements are true or false, and *briefly* justify why. No partial credit will be given for unjustified answers.

(1) Assume S is a finite probability space with probability function P. For any two events A and B, $P(A \cup B) \leq P(A) + P(B)$.
(2) The cartesian product of two uncountable sets is uncountable.
(3) Any sequence of 42 distinct real numbers contains a monotone subsequence of length 8.
(4) For any natural numbers n and k, there are $\frac{n!}{(n-k)!k!}$ subsets of $[n]$ with k elements.
(5) If $x_0 \equiv x_1 \bmod n$ and $y_0 \equiv y_1 \bmod n$, then $x_0^{y_0} \equiv x_1^{y_1} \bmod n$.

Problem 3 (30 points)

Prove any two of the following three results seen in class or in your assignment. Clearly mark which two you have chosen.

(1) Fermat's little theorem: For p a prime and x an integer, $x^p \equiv x \bmod p$.
(2) Pascal's formula: For natural numbers k and n, $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.
(3) $\mathbb{N} \times \mathbb{N}$ is countable.

Problem 4 (15 points)

(1) Compute the last (i.e. least significant) digit of 2015^{2014} in base 11.
(2) Assume n is a positive integer which is divisible by 7. Pick a natural number $k \in [n]$ uniformly at random. What is the probability that the remainder of the division of k by 7 is 1?

Problem 5 (15 points)

Assume A and B are sets, $f : A \to A$, $g : A \to B$ are functions. Prove or disprove:

(1) If f is an injection and g is a bijection, then $g \circ f$ is a surjection.
(2) If B is finite, f is an injection, and g is a bijection, then $g \circ f$ is a surjection.

Extra credit (20 points)

For $A, B \subseteq \mathbb{R}$, a function $f : A \to B$ is *increasing* if for any x and y in A, $x < y$ implies $f(x) < f(y)$.

For example, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 3$ is increasing, but $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$ is not ($-1 < 0$ but $(-1)^2 = 1 > 0^2 = 0$).
Show that there is no increasing bijection from $\mathbb{N}$ to $\mathbb{Z}$.