Abstract elementary classes categorical in a high-enough limit cardinal[1]

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Introduction

Observation

Let λ be an uncountable cardinal.

- There is a unique $\mathbb{Q}$-vector space with cardinality λ.
- There is a unique algebraically closed field of characteristic zero with cardinality λ.

Definition (Łoś, 1954)

A class of structure (or a sentence, or a theory) is *categorical in* λ if it has exactly one model of cardinality λ (up to isomorphism).

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Question

If K is "reasonable", can we say something about the class of cardinals in which K is categorical?

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Theorem (Morley, 1965)

Let K be the class of models of a countable first-order theory. If K is categorical in *some* $\lambda \geq \aleph_1$, then K is categorical in *all* $\lambda' \geq \aleph_1$.

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Conjecture (Shelah, 197?)

If an $\mathbb{L}_{\omega_1,\omega}$ sentence is categorical in *some* $\lambda \geq \beth_{\omega_1}$, then it is categorical in *all* $\lambda' \geq \beth_{\omega_1}$.

Eventual version for AECs: If an AEC is categorical in *some* high-enough cardinal, then it is categorical in *all* high-enough cardinal.

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Shelah's eventual categoricity conjecture in universal classes

Theorem (V.)

Let ψ be a *universal* $\mathbb{L}_{\omega_1,\omega}$-sentence. If ψ is categorical in *some* $\lambda \geq \beth_{\beth_{\omega_1}}$, then ψ is categorical in *all* $\lambda' \geq \beth_{\beth_{\omega_1}}$.

This has a natural generalization to uncountable vocabularies using the framework of universal classes (classes closed under isomorphisms, substructures, and unions of chains). Set $h(\mu) := \beth_{(2^\mu)^+}$:

Theorem (V.)

Let K be a universal class. If K is categorical in *some* $\lambda \geq \beth_{h(|\tau(K)|+\aleph_0)}$, then K is categorical in *all* $\lambda' \geq \beth_{h(|\tau(K)|+\aleph_0)}$.

Two general categoricity transfers

Let **K** be an AEC.

Theorem (Model theoretic version, V.)

Assume that **K** has amalgamation, is χ-tame, and has primes over sets of the form *Ma*.

If **K** is categorical in *some* $\lambda \geq h(\chi)$, then **K** is categorical in *all* $\lambda' \geq h(\chi)$.

Corollary (Large cardinal version, V.)

Let $\kappa > \mathrm{LS}(\mathbf{K})$ be strongly compact. Assume that **K** has primes over sets of the form *Ma*.

If **K** is categorical in *some* $\lambda \geq h(\kappa)$, then **K** is categorical in *all* $\lambda' \geq h(\kappa)$.

Questions to explore

- How do these results compare to earlier ones?
- What is the role of large cardinals?
- How is the "primes" hypothesis used?
- How does being a universal class help?
- What classes have primes?

Amalgamation

Definition

An AEC $\mathbf{K}$ has *amalgamation* if whenever $M_0 \leq_{\mathbf{K}} M_\ell$, $\ell = 1, 2$, there exists $N \in \mathbf{K}$ and $f_\ell : M_\ell \underset{M_0}{\longrightarrow} N$.

$$\begin{tikzcd} M_1 \arrow[r, dotted, "f_1"'] & N \\ M_0 \arrow[u] \arrow[r] & M_2 \arrow[u, dotted, "f_2"'] \end{tikzcd}$$

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Amalgamation can fail in general AECs, even in universal classes.

Theorem (Kolesnikov and Lambie-Hanson, 2015)

For every $\alpha < \omega_1$, there exists a universal class in a countable vocabulary that has amalgamation up to $\beth_\alpha$ but fails amalgamation starting at $\beth_{\omega_1}$.

Orbital (Galois) types and tameness

Definition

For $\mathbf{K}$ an AEC:

- (Shelah) $(a, M_0, M_1) E_{\text{at}} (b, M_0, M_2)$ if there exists N with:

$$\begin{tikzcd} M_1 \arrow[r, dotted, "f_1"'] & N \\ M_0 \arrow[u, "{[a]}"] \arrow[r, "{[b]}"'] & M_2 \arrow[u, dotted, "f_2"'] \end{tikzcd}$$

  and $f_1(a) = f_2(b)$. Let E be the transitive closure of E_{at} and $\mathbf{tp}(a/M_0; M_1) := [(a, M_0, M_1)]_E$.

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  and $f_1(a) = f_2(b)$. Let E be the transitive closure of E_{at} and $\mathbf{tp}(a/M_0; M_1) := [(a, M_0, M_1)]_E$.
- (Grossberg-VanDieren) For $\chi \geq \mathrm{LS}(\mathbf{K})$, $\mathbf{K}$ is χ*-tame* if whenever $\mathbf{tp}(a/M_0; M_1) \neq \mathbf{tp}(b/M_0; M_2)$, there exists $N \leq_{\mathbf{K}} M_0$ with $\|N\| \leq \chi$ and $\mathbf{tp}(a/N; M_1) \neq \mathbf{tp}(b/N; M_2)$.

Primes

Definition (Shelah)

An AEC $\mathbf{K}$ *has primes* if for any (orbital) type p over M_0, there exists a triple (a, M_0, M_1) such that $p = \mathbf{tp}(a/M_0; M_1)$ and whenever $p = \mathbf{tp}(b/M_0; M_2)$, there exists $f : M_1 \underset{M_0}{\longrightarrow} M_2$ with $f(a) = b$.

(in the diagram below, $a = b$):

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In universal classes the closure of $M_0 a$ to a substructure gives a prime model over $M_0 a$.

Earlier approximations to SECC

Theorem

Let $\mathbf{K}$ be an AEC with amalgamation.

- (Shelah 1999) If $\mathbf{K}$ is categorical in *some* successor $\lambda \geq \beth_{h(\mathrm{LS}(\mathbf{K}))}$, then $\mathbf{K}$ is categorical in *all* $\lambda' \in [\beth_{h(\mathrm{LS}(\mathbf{K}))}, \lambda]$.

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- (Grossberg-VanDieren 2006) If $\mathbf{K}$ is χ-tame and categorical in *some* successor $\lambda > \chi^+$, then $\mathbf{K}$ is categorical in *all* $\lambda' \geq \lambda$.

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- (Grossberg-VanDieren 2006) If $\mathbf{K}$ is χ-tame and categorical in *some* successor $\lambda > \chi^+$, then $\mathbf{K}$ is categorical in *all* $\lambda' \geq \lambda$.
- (Shelah 2009; assuming an unpublished claim) Assume $2^\lambda < 2^{\lambda^+}$ for all cardinals λ. If $\mathbf{K}$ is categorical in *some* $\lambda \geq h(\aleph_{\mathrm{LS}(\mathbf{K})^+})$, then $\mathbf{K}$ is categorical in *all* $\lambda' \geq h(\aleph_{\mathrm{LS}(\mathbf{K})^+})$.

Earlier approximations to SECC, with large cardinals

Theorem (Makkai-Shelah, Boney)

If $\kappa > \mathrm{LS}(\mathbf{K})$ is strongly compact, then $\mathbf{K}$ is $(< \kappa)$-tame (in fact fully $(< \kappa)$-tame and short).

Theorem (Makkai-Shelah, Boney)

If $\kappa > \mathrm{LS}(\mathbf{K})$ is strongly compact and $\mathbf{K}$ is categorical in *some* $\lambda \geq h(\kappa)$, then $\mathbf{K}_{\geq \kappa}$ has amalgamation.

Therefore SECC *with categoricity in a successor* follows from the existence of a proper class of strongly compact cardinals.

Categoricity in universal classes

Theorem (V.)

If a universal class K is categorical in *some* $\lambda \geq \beth_{h(|\tau(K)|+\aleph_0)}$, then K is categorical in *all* $\lambda' \geq \beth_{h(|\tau(K)|+\aleph_0)}$.

1. Does *not* assume that the categoricity cardinal is a successor.

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2. Does *not* assume amalgamation or tameness.

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2. Does *not* assume amalgamation or tameness.
3. Does *not* use large cardinals.

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4. Does *not* assume any cardinal arithmetic hypotheses (or any unpublished claims). Is proven entirely in ZFC.

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We *do* assume that K is a universal class.

"Niceness" should follow from categoricity

Question (Grossberg)

Does eventual amalgamation follow from high-enough categoricity?

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Does the eventual existence of primes follow from high-enough categoricity?

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Question (Grossberg-VanDieren)

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Question

Does the eventual existence of primes follow from high-enough categoricity?

In the presence of large cardinals, the first questions/conjectures become theorems, sometimes with (too) short proofs! The third is open, even with large cardinals.

They also become theorems in universal classes.

Categoricity in universal classes, step one

Theorem (V.)

Let K be a universal class. If K is categorical in *some* $\lambda \geq \beth_{h(|\tau(K)|+\aleph_0)}$, then there exists an ordering $\leq$ such that:

1. $\mathbf{K}^* := (K, \leq)$ is an AEC with $\chi := \mathrm{LS}(\mathbf{K}^*) < h(|\tau(K)| + \aleph_0)$.
2. $\mathbf{K}^*_{\geq \chi}$ has amalgamation, is χ-tame, and has primes.

This uses Shelah's classification theory for universal classes, and more.

Shelah's eventual categoricity conjecture for universal classes then follows from the categoricity transfer for tame AECs with amalgamation and primes.

Justifying the "primes" hypothesis

Theorem (V.)

Let **K** be a χ-tame AEC with amalgamation and primes.

If **K** is categorical in *some* $\lambda \geq h(\chi)$, then **K** is categorical in *all* $\lambda' \geq h(\chi)$.

This gives another proof of (the eventual version of) Morley's theorem, Shelah's generalization to uncountable languages, and the categoricity conjecture for homogeneous model theory.

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There is also a converse:

Theorem (V.)

Let **K** be a <u>fully χ-tame and short AEC with amalgamation</u>.

If **K** is categorical in *all* $\lambda' \geq h(\chi)$, then $\mathbf{K}_{\geq h(\chi)}$ has primes.

Justifying the "primes" hypothesis

Definition (Baldwin-Shelah)

An AEC $\mathbf{K}$ *admits intersections* if for any $N \in \mathbf{K}$ and $A \subseteq |N|$, the set

$$\mathrm{cl}^N(A) := \bigcap \{|M| : M \leq_{\mathbf{K}} N, A \subseteq |M|\}$$

is the universe of a $\leq_{\mathbf{K}}$-substructure of N.

Universal classes admit intersections. Any AEC which admits intersections has primes.

A proof sketch

Let **K** be a χ-tame AEC with amalgamation and primes. Let $\mu < \lambda$ both be "high-enough" categoricity cardinals. We show that **K** is categorical in μ^+.

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1. **K** is "good" in μ.
2. AFSOC that **K** is *not* categorical in μ^+. Then a type *p* over a model of size μ is omitted by a model of size μ^+.

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2. AFSOC that $\mathbf{K}$ is *not* categorical in μ^+. Then a type p over a model of size μ is omitted by a model of size μ^+.
3. $\mathbf{K}_{\neg p}$, the class of models omitting p, is an AEC and it is "good" in μ. Further, $\mathbf{K}_{\neg p}$ is tame and has primes.

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4. Goodness transfers up (uses tameness and primes): $\mathbf{K}_{\neg p}$ is "good" also above μ.

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4. Goodness transfers up (uses tameness and primes): $\mathbf{K}_{\neg p}$ is "good" also above μ.
5. By "goodness", $\mathbf{K}_{\neg p}$ has a model of cardinality λ.
6. This contradicts categoricity in λ (the model there is saturated).

References

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