Stability theory for concrete categories

Sebastien Vasey

Harvard University

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University of Cambridge

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The notation is due to Erdős and Rado. It means: for any set X with at least n elements and any coloring F of the unordered pairs from X in two colors, there exists $H \subseteq X$ with $|H| = k$ so that F is constant on the pairs from H (we call H a *homogeneous set* for F).

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If $k = 3$, $n = 6$ suffices. If $k = 5$, the optimal value of n is not known.

An infinite variation on the puzzle

If an infinite number of students come to a party, then infinitely-many all know each other or infinitely-many all do not know each other.

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The theorem does *not* say that $|X| = |H|$: it does *not* rule out a party with uncountably-many students where all friends/strangers groups (= homogeneous sets) are countable.

Ramsey's dream

For any infinite cardinal λ, if λ students come to a party, then there is a group of λ-many that all know each other or a group of λ-many that all do not know each other. That is:

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This is wrong for most cardinals λ.

Counterexamples to Ramsey's dream

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In the reals, a countable set allows one to distinguish uncountably-many points. There are however many structures where this is not the case.

Ramsey's dream in the complex field

Proposition

If F is a coloring of the unordered pairs of complex numbers in two colors *such that* $F(\{f(x), f(y)\}) = F(\{x, y\})$ *for any field automorphism* f *of* $\mathbb{C}$, then F has a homogeneous set of cardinality $|\mathbb{C}|$.

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This proves $|\mathbb{C}| \to |\mathbb{C}|_2$ but "relativized to $\mathbb{C}$" (for colorings preserved by automorphisms).

Types

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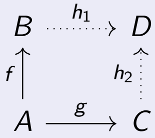
Definition

Given a concrete category **K** with amalgamation and an object A of **K**, a *type over* A is just a pair $(x, A \xrightarrow{f} B)$, with $x \in B$. Two types $(x, A \xrightarrow{f} B)$, $(y, A \xrightarrow{g} C)$ are considered *the same* if there exists maps h_1, h_2 so that $h_1(x) = h_2(y)$ and the following diagram commutes:



Types in fields, linear orders, and graphs

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In the category of linear orders, there are $|\mathbb{R}|$ types over $\mathbb{Q}$. In general, types correspond to Dedekind cuts.

In the category of graphs with induced subgraph embeddings, there are at least $2^{|V(G)|}$ types over any graph G.

Definition (Stability)

A concrete category **K** is *stable in* λ if there are at most λ-many types over any object of cardinality λ. *Stable* means stable in an unbounded class.

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- (Kucera and Mazari-Armida) The category of R-modules with pure embeddings is always stable, and stable in all cardinals if and only if R is pure-semisimple.

Ramsey's dream in stable AECs

Theorem (V.)

If **K** is an abstract elementary class with amalgamation and **K** is stable in λ, then:

$$\lambda^+ \xrightarrow{\mathbf{K}} \left(\lambda^+\right)_\lambda$$

Here λ^+ is the cardinal right after λ.

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Here λ^+ is the cardinal right after λ.

The partition notation means that given objects $A \to B$ in **K** with $|A| = \lambda$, $|B| = \lambda^+$, if F is a coloring of pairs from B in λ-many colors so that any two pairs with the same type over A have the same color, then we can find a homogeneous set for F of cardinality λ^+.

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Definition (Shelah, late 1970s)

An abstract elementary class (AEC) is a concrete category **K** satisfying the following conditions:

- All morphisms are concrete monomorphisms (injections).
- **K** has concrete directed colimits (also known as direct limits – basically closure under unions of increasing chains).
- (Smallness condition) Every object is a directed colimit of a fixed set of "small" subobjects.

Examples of abstract elementary classes

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Any AEC is an accessible category: a category with all sufficiently directed colimits satisfying a certain smallness condition.

Abstract elementary classes and logic

A *first-order formula* is a statement like $(\forall x \exists y)(x \cdot y = 1)$.

For any list T of first-order formulas, the category $\mathrm{Mod}(T)$ of models of T forms an AEC (the morphisms are the functions preserving all formulas).

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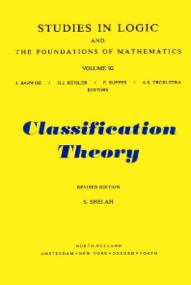
We will call such a category a *first-order class*. It is one of the basic objects of study in model theory.

Stability theory was developped for first-order classes first, by Saharon Shelah.



Beyond first-order classes

First-order classes are important, because of the *compactness theorem*: if all finite subsets of a given theory have a model, then the whole theory has a model. This is powerful (one can use it to build models for nonstandard analysis) but means that many interesting categories are not first-order.

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Also, the *morphisms* of first-order classes are not so natural.

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One goal of the research presented here is **to develop a general framework for the parts of model theory that are "category-theoretic".**

Shelah's eventual categoricity conjecture

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Thus it seems any AEC with a "perfect theory of dimension" should have unique objects of each high-enough cardinality. Morley (1965) proved a sort of converse for first-order classes, and Shelah proposed this should generalize:

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The only known way to prove such statements is via stability theory.

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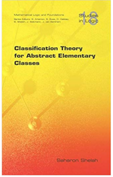
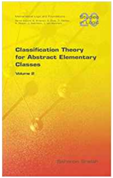
Partial approximations before my thesis include: Shelah 1983, Makkai-Shelah 1990, Shelah 1999, Shelah-Villaveces 1999, VanDieren 2006, Grossberg-VanDieren 2006, Shelah 2009, Hyttinen-Kesälä 2011, Boney 2014.





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Theorem (V. 2019)

Assuming the GCH, Shelah's eventual categoricity conjecture is true for AECs with amalgamation. In this case one can list all possibilities for the class of cardinals in which the category has a unique object.

Stability and order

Theorem (V. 2016, Boney)

A tame AEC $\mathbf{K}$ with amalgamation is stable if and only if it does not have the "order property": any faithful functor $\mathbf{Lin} \xrightarrow{F} \mathbf{K}$ factors through the forgetful functor.

$$\begin{array}{ccc} \mathbf{Lin} & \xrightarrow{F} & \mathbf{K} \\ \downarrow U & \nearrow & \\ \mathbf{Set} & & \end{array}$$

Order in graphs: an intermission

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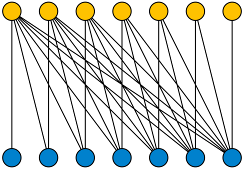
It is given by a half graph: for any linear ordering L, consider the bipartite graph on $L \sqcup L$ where we put an edge from i to j if only if $i \leq j$ (the picture below is for $L = \{1, 2, 3, 4, 5, 6, 7\}$):



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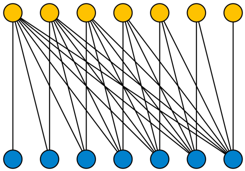

Graphs omitting half graphs are studied in finite combinatorics too (Malliaris-Shelah, *Regularity lemmas for stable graphs*. TAMS 2014).

Stable independence

The proofs of the eventual categoricity conjecture and of the partition theorem $\lambda^+ \xrightarrow{\mathbf{K}} (\lambda^+)_\lambda$ involve describing what it means for a type to be "determined" over a small base. This is called forking in the first-order context, and is the key tool developped by Shelah in his classification theory book. It generalizes algebraic independence in fields.

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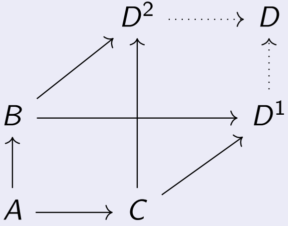
Unfortunately Shelah's definition is syntactic, hard to describe, and some properties depend on compactness. With my collaborators, we found a completely category-theoretic definition.

Definition (Equivalence of amalgam)

Consider a diagram: $B \leftarrow A \rightarrow C$.

Two amalgams $B \rightarrow D^1 \leftarrow C$, $B \rightarrow D^2 \leftarrow C$ of this diagram are *equivalent* if there exists D and arrows making the following diagram commute:



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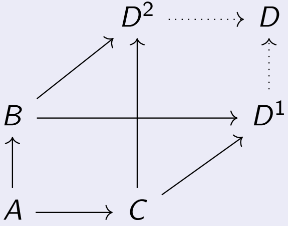

Example: in $\mathbf{Set}_{mono}$, $\{0\}$ and $\{1\}$ have two non-equivalent amalgams over $\emptyset$: $\{0, 1\}$ and $\{1\}$ (with the expected morphisms).

Definition (Stable independence; Lieberman-Rosický-V., 2019)

A *stable independence notion* is a class of squares (called *independent squares*, marked with $\mathrel{\raise0.2ex\hbox{\ooalign{\hidewidth$\vert$\hidewidth\cr\raise-0.9ex\hbox{$\smile$}}}}$) such that:

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2. Existence: any span can be amalgamated to an independent square.
3. Uniqueness: any two *independent* amalgam of the same span are equivalent.
4. Symmetry:

$$\begin{array}{ccc} B & \longrightarrow & D \\ \uparrow & \downarrow & \uparrow \\ A & \longrightarrow & C \end{array} \quad \Rightarrow \quad \begin{array}{ccc} C & \longrightarrow & D \\ \uparrow & \downarrow & \uparrow \\ A & \longrightarrow & B \end{array}$$

Definition (stable independence notion - continued)

5. Transitivity:

$$
\begin{array}{ccccc}
B & \longrightarrow & D & \longrightarrow & F \\
\uparrow & \perp & \uparrow & \perp & \uparrow \\
A & \longrightarrow & C & \longrightarrow & E
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6. Accessibility: the category whose objects are arrows and whose morphisms are independent squares is accessible. This implies that any arrow can be "filtered" in an independent way:

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\uparrow & \overset{|}{\smile} & \uparrow \\
A_i & \cdots\cdots\rightarrow & B_i
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A category with directed colimits (in particular an AEC) has *at most one* stable independence notion.

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Theorem (Lieberman-Rosický-V. 2019)

An AEC with a stable independence notion has amalgamation, is tame, and is stable.

Certain converses are true too (for example in first-order classes, or assuming large cardinals).

Stable independence and cofibrant generation

Theorem (Lieberman-Rosický-V.)

Let $\mathcal{K}$ be an accessible cocomplete category (like the category of R-modules with homomorphisms). Let $\mathcal{M}$ be a class of morphisms of $\mathcal{K}$ satisfying reasonable closure properties (like the monos, or the pure monos).

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Then the subcategory of $\mathcal{K}$ with only morphisms from $\mathcal{M}$ has stable independence if and only if $\mathcal{M}$ is cofibrantly generated (i.e. can be generated from a small subclass using transfinite compositions, pushouts, and retracts).

New examples of stable independence

Corollary (Lieberman-Rosický-V.)

1. The AEC of flat R-modules with flat morphisms (more generally, any AEC of "roots of Ext") has stable independence.
2. Any Grothendieck topos restricted to regular monos has stable independence.
3. Any Grothendieck abelian category restricted to monos has stable independence.
4. Any Cisinski model category restricted to monos has stable independence.

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Some directions for future work:

- What are applications of these connections? Ongoing work: a simple proof of a theorem of Makkai-Rosický on existence of pseudopullback for combinatorial categories.
- Where else does stable independence occur?
- Develop a systematic theory of higher-dimensional stable independence.

Thank you!

Some references:

- Sebastien Vasey, *Shelah's eventual categoricity conjecture in universal classes: part II*, Selecta Mathematica **23** (2017), no. 2, 1469–1506.
- Michael Lieberman, Jiří Rosický, and Sebastien Vasey, *Forking independence from the categorical point of view*, Advances in Mathematics **346** (2019), 719–772.
- Sebastien Vasey, *The categoricity spectrum of large abstract elementary classes with amalgamation*, Selecta Mathematica **25** (2019), no. 5, 65 (51 pages).
- Saharon Shelah and Sebastien Vasey, *Categoricity and multidimensional diagrams*, arXiv:1805.0629.
- Michael Lieberman, Jiří Rosický, and Sebastien Vasey, *Weak factorization systems and stable independence*, arXiv:1904.05691.
- Sebastien Vasey, *Accessible categories, set theory, and model theory: an invitation*, arXiv:1904.11307.