Stability theory for concrete categories

Sebastien Vasey

Harvard University

January 20, 2019
IST Austria

A puzzle

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The notation is due to Erdős and Rado. It means: for any set X with at least n elements and any coloring F of the unordered pairs from X in two colors, there exists $H \subseteq X$ with $|H| = k$ so that F is constant on the pairs from H (we call H a *homogeneous set* for F).

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If $k = 3$, $n = 6$ suffices. If $k = 5$, the optimal value of n is not known.

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The theorem does *not* say that $|X| = |H|$: it does *not* rule out a party with uncountably-many students where all friends/strangers groups ($=$ homogeneous sets) are countable.

Ramsey's dream

For any infinite cardinal λ, if λ students come to a party, then there is a group of λ-many that all know each other or a group of λ-many that all do not know each other. That is:

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This is wrong for most cardinals λ.

The Sierpiński coloring

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Proof.

Fix a well-ordering $\lhd$ of the reals. Set $F(\{x, y\}) = 1$ when $x < y$ iff $x \lhd y$, and $F(\{x, y\}) = 0$ otherwise (F is called the *Sierpiński coloring*). Assume for a contradiction H is an uncountable homogeneous set for F. Without loss of generality, for $x, y \in H$, $x < y$ if and only if $x \lhd y$. As $\lhd$ is a well-ordering, each $x \in H$ has an immediate successor x' in H. Find a rational r_x between x and x'. Then $x \to r_x$ is an injection of H (uncountable) into the rationals (countable), contradiction. □

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The Sierpiński coloring relies on a well-ordering of the reals. Is there a more "natural" counterexample?

A counterexample with an infinite number of colors

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In the reals, a countable set allows one to distinguish uncountably-many points. There are however many structures where this is not the case.

Ramsey's dream in the complex field

Proposition

If F is a coloring of the unordered pairs of complex numbers in two colors *such that* $F(\{f(x), f(y)\}) = F(\{x, y\})$ *for any field automorphism* f *of* $\mathbb{C}$, then F has a homogeneous set of cardinality $|\mathbb{C}|$.

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Proof.

Any transcendence basis for $\mathbb{C}$ does the job. □

This proves $|\mathbb{C}| \to |\mathbb{C}|_2$ but "relativized to $\mathbb{C}$" (for colorings preserved by automorphisms).

Types

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Definition

Given a concrete category $\mathbf{K}$ with amalgamation and an object A of $\mathbf{K}$, a *type over* A is just a pair $(x, A \xrightarrow{f} B)$, with $x \in B$. Two types $(x, A \xrightarrow{f} B)$, $(y, A \xrightarrow{g} C)$ are considered *the same* if there exists maps h_1, h_2 so that $h_1(x) = h_2(y)$ and the following diagram commutes:

$$\begin{array}{ccc} B & \overset{h_1}{\dashrightarrow} & D \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle h_2} \\ A & \xrightarrow{g} & C \end{array}$$

Types in fields, linear orders, and graphs

Essentially, one can think of types over a fixed base A as the orbits of an automorphism group fixing A.

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In the category of linear orders, there are $|\mathbb{R}|$ types over $\mathbb{Q}$. In general, types correspond to Dedekind cuts.

In the category of graphs with induced subgraph embeddings, there are at least $2^{|V(G)|}$ types over any graph G.

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- (Kucera and Mazari-Armida) The category of R-modules with pure embeddings is always stable, and stable in all cardinals if and only if R is pure-semisimple.

Ramsey's dream in stable AECs

Theorem (V.)

If $\mathbf{K}$ is an abstract elementary class with amalgamation and $\mathbf{K}$ is stable in λ, then:

$$\lambda^+ \xrightarrow{\mathbf{K}} (\lambda^+)_\lambda$$

Here λ^+ is the cardinal right after λ.

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Here λ^+ is the cardinal right after λ.

The partition notation means that given objects $A \to B$ in **K** with $|A| = \lambda$, $|B| = \lambda^+$, if F is a coloring of pairs from B in λ-many colors so that any two pairs with the same type over A have the same color, then we can find a homogeneous set for F of cardinality λ^+.

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Definition (Shelah, late 1970s)

An abstract elementary class (AEC) is a concrete category **K** satisfying the following conditions:

- All morphisms are concrete monomorphisms (injections).
- **K** has concrete directed colimits (also known as direct limits – basically closure under unions of increasing chains).
- (Smallness condition) Every object is a directed colimit of a fixed set of "small" subobjects.

Examples of abstract elementary classes

All the categories mentioned before are AECs.

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Any AEC is an accessible category: a category with all sufficiently directed colimits satisfying a certain smallness condition.

Abstract elementary classes and logic

A *first-order formula* is a statement like $(\forall x \exists y)(x \cdot y = 1)$.

For any list T of first-order formulas, the category $\text{Mod}(T)$ of models of T forms an AEC (the morphisms are the functions preserving all formulas).

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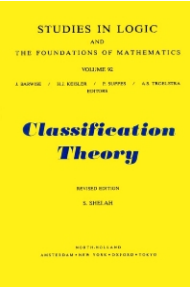
We will call such a category a *first-order class*. It is one of the basic objects of study in model theory.

Stability theory was developped for first-order classes first, by Saharon Shelah.



Beyond first-order classes

First-order classes are important, because of the *compactness theorem*: if all finite subsets of a given theory have a model, then the whole theory has a model. This is powerful (one can use it to build models for nonstandard analysis) but means that many interesting categories are not first-order.

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Also, the *morphisms* of first-order classes are not so natural.

Example

The category of fields is not first-order because the embedding $\mathbb{Q} \to \mathbb{R}$ does not preserve the formula $(\exists x)(x \cdot x = 2)$.

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One goal of the research presented here is **to develop a general framework for the parts of model theory that are "category-theoretic".**

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Thus it seems any AEC with a "perfect theory of dimension" should have unique objects of each high-enough cardinality. Morley (1965) proved a sort of converse for first-order classes, and Shelah proposed this should generalize:

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The only known way to prove such statements is via stability theory.

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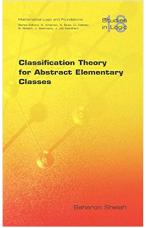
Partial approximations before
my thesis include: Shelah 1983, Makkai-Shelah 1990, Shelah 1999, Shelah-Villaveces 1999, VanDieren 2006, Grossberg-VanDieren 2006, Shelah 2009, Hyttinen-Kesälä 2011, Boney 2014.



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Theorem (V. 2019)

Assuming the GCH, Shelah's eventual categoricity conjecture is true for AECs with amalgamation. In this case one can list all possibilities for the class of cardinals in which the category has a unique object.

Locality of types

Definition (Grossberg-VanDieren 2006)

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Example

For any finite $A \subseteq (0, 1)$ there is an automorphism of $(\mathbb{R}, <)$ sending 1 to 2 and fixing A. However there is *no* such automorphism sending 1 to 2 and fixing the whole open interval $(0, 1)$.

How prevalent is tameness?

Theorem (V. 2019)

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Earlier Boney (2014) had shown that tameness follows from a large cardinal axiom, and always holds in universal AECs.

It is now known that many concrete examples are tame (including the ones from the beginning of the talk).

Stability and order

Theorem (V. 2016, Boney)

A tame AEC $\mathbf{K}$ with amalgamation is stable if and only if it does not have the "order property": any faithful functor $\mathbf{Lin} \xrightarrow{F} \mathbf{K}$ factors through the forgetful functor.

$$\begin{array}{ccc} \mathbf{Lin} & \xrightarrow{F} & \mathbf{K} \\ \downarrow{\scriptstyle U} & \nearrow & \\ \mathbf{Set} & & \end{array}$$

Order in graphs: an intermission

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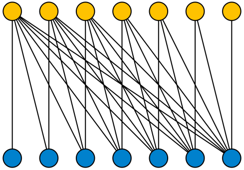
It is given by a half graph: for any linear ordering L, consider the bipartite graph on $L \sqcup L$ where we put an edge from i to j if only if $i \leq j$ (the picture below is for $L = \{1, 2, 3, 4, 5, 6, 7\}$):



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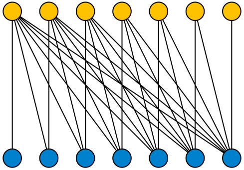

Graphs omitting half graphs are studied in finite combinatorics too (Malliaris-Shelah, *Regularity lemmas for stable graphs*. TAMS 2014).

Stable independence

The proofs of the eventual categoricity conjecture and of the partition theorem $\lambda^+ \xrightarrow{\mathbf{K}} (\lambda^+)_\lambda$ involve describing what it means for a type to be "determined" over a small base. This is called forking in the first-order context, and is the key tool developped by Shelah in his classification theory book. It generalizes algebraic independence in fields.

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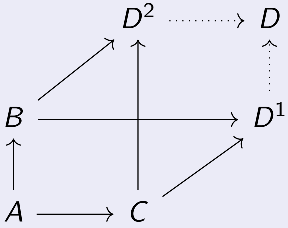
Unfortunately Shelah's definition is syntactic, hard to describe, and some properties depend on compactness. With my collaborators, we found a completely category-theoretic definition.

Definition (Equivalence of amalgam)

Consider a diagram: $B \leftarrow A \rightarrow C$.

Two amalgams $B \rightarrow D^1 \leftarrow C$, $B \rightarrow D^2 \leftarrow C$ of this diagram are *equivalent* if there exists D and arrows making the following diagram commute:



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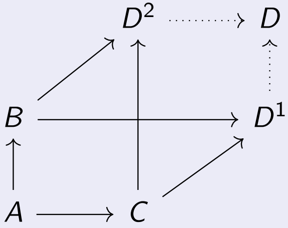

Example: in $\mathbf{Set}_{mono}$, $\{0\}$ and $\{1\}$ have two non-equivalent amalgams over $\emptyset$: $\{0, 1\}$ and $\{1\}$ (with the expected morphisms).

Definition (Stable independence; Lieberman-Rosický-V., 2019)

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2. Existence: any span can be amalgamated to an independent square.
3. Uniqueness: any two *independent* amalgam of the same span are equivalent.
4. Symmetry:

$$\begin{array}{ccc} B & \longrightarrow & D \\ \uparrow & \stackrel{\mid}{\smile} & \uparrow \\ A & \longrightarrow & C \end{array} \quad \Rightarrow \quad \begin{array}{ccc} C & \longrightarrow & D \\ \uparrow & \stackrel{\mid}{\smile} & \uparrow \\ A & \longrightarrow & B \end{array}$$

Definition (stable independence notion - continued)

5. Transitivity:

$$
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\uparrow & \perp & \uparrow & \perp & \uparrow \\
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6. Accessibility: the category whose objects are arrows and whose morphisms are independent squares is accessible. This implies that any arrow can be "filtered" in an independent way:

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\vdots\!\uparrow & \underset{\smile}{\perp} & \vdots\!\uparrow \\
A_i & \cdots\cdots\to & B_i
\end{array}
$$

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Theorem (Lieberman-Rosický-V. 2019)

An AEC with a stable independence notion has amalgamation, is tame, and is stable.

Certain converses are true too (for example in first-order classes, or assuming large cardinals).

Stable independence and cofibrant generation

Theorem (Lieberman-Rosický-V.)

Let $\mathcal{K}$ be an accessible cocomplete category (like the category of R-modules with homomorphisms). Let $\mathcal{M}$ be a class of morphisms of $\mathcal{K}$ satisfying reasonable closure properties (like the monos, or the pure monos).

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Then the subcategory of $\mathcal{K}$ with only morphisms from $\mathcal{M}$ has stable independence if and only if $\mathcal{M}$ is cofibrantly generated (i.e. can be generated from a small subclass using transfinite compositions, pushouts, and retracts).

New examples of stable independence

Corollary (Lieberman-Rosický-V.)

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Summary and future work

We have seen several ways to think of stability:

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- Cofibrant generation in abstract homotopy theory ("morphisms being generated by a small set").

Some directions for future work:

- What are applications of these connections? Ongoing work: a simple proof of a theorem of Makkai-Rosický on existence of pseudopullback for combinatorial categories.
- Where else does stable independence occur?
- Develop a systematic theory of higher-dimensional stable independence.

Thank you!

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